INTRINSIC GEOMETRY
OF SURFACES

by
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Two-dimensional Manifolds of Bounded Curvature
(Foundations of the intrinsic geometry of surfaces)

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INTRODUCTION

The present monograph is devoted to a systematic presentation of the intrinsic geometry of nonregular surfaces. The work of Gauss on intrinsic geometry deals only with regular surfaces. The Riemannian geometry of abstract metrized manifolds is also restricted to those admitting a regularly defined metric. The construction of the geometry of nonregular convex surfaces was achieved by A. D. Aleksandrov in [5]. In the concluding section of that monograph a program was indicated for the construction of the intrinsic geometry of general surfaces. What was wanted was a system of concepts and methods which would be applicable equally to the investigation of the intrinsic geometry of regular surfaces and of two-dimensional Riemannian manifolds, polyhedra and polyhedral evolutes, general convex surfaces, and also to the investigation of the widest possible class of nonconvex nonregular surfaces and metrized two-dimensional manifolds.

In the papers [6], [14], [17], [46], this program was worked out in more detail. The principal results were communicated without proofs. Under some additional hypotheses, A. D. Aleksandrov succeeded in investigating two-dimensional metric spaces by geometric methods, and in particular in introducing into them the concepts of shortest and geodesic curves, angles, integral curvature, area, direction and rotation of curves, and in establishing the most important properties of these concepts.

These spaces have been called “two-dimensional manifolds of bounded curvature.” They are the natural closure of the class of Riemannian spaces. This extended class is obtained by adding to the two-dimensional Riemannian spaces all two-dimensional metrized manifolds whose metric in the neighborhood of each point may be uniformly approximated by Riemannian metrics such that the integrals of the absolute values of the Gaussian curvatures are uniformly bounded.

The types of spaces and surfaces enumerated above belong to this class. At the same time the above class of surfaces admits common methods of investigation. These extend techniques developed by A. D. Aleksandrov for convex surfaces. Among them are the following.

1. An axiomatic method, starting from the definition of such spaces by means of a minimal choice of the properties of their metrics.

2. An approximative method, based on approximation by polyhedral or Riemannian metrics. This method makes use of theorems on the
possibility of appropriate approximations of the spaces themselves and of figures in them by simpler spaces and figures, and also of general theorems on the connection between the numerical characteristics of the converging figures and of the limiting figure.

3. A synthetic method based on geometric constructions in such spaces and a study in them of curves, triangles and other figures. This method makes use in particular, of the comparison of figures in such a space with similar figures on the plane and of methods such as the cutting and pasting of new spaces from pieces of existing spaces.

The results obtained, for all their generality, retain their geometric intuitiveness. The fact that the class of spaces in question is closed makes it possible to state and solve extremal problems in that class in a natural way. The basic restriction adopted turns out to be completely natural. The integral curvature characterizes the deviation of the intrinsic geometry of the surface from Euclidean geometry, and the restriction on the integral curvature makes it in fact possible to retain the basic integral concepts of classical differential geometry.

The actual construction of a theory of two-dimensional manifolds of bounded curvature is the object of the present work.

An exposition beginning with a small number of initial axioms requires a gradual accumulation of facts. Analogously, if we make use only of the possibility of approximation by polyhedral metrics, we need information about those polyhedral metrics. Therefore the exposition is carried out cyclically. Certain results are established at first in less than their full extent or in special cases, and later they are extended to more general, definite results.

A large number of later papers by Soviet geometers deal with the material of this monograph. The authors intend to prepare for publication a collection of papers in the directions indicated in §6 of Chapter I.
Chapter I

The Simplest Concepts and the Object of
the Investigation

This chapter contains a survey of the fundamental concepts and definitions of the spaces which we shall investigate. In contrast to the practice in the following chapters, the assertions are formulated without proofs. In §6 we enumerate some investigations on two-dimensional manifolds of bounded curvature which are not included in this monograph.

1. The intrinsic metric of a space.

1. Metric spaces. In what follows we shall consider only metric spaces. For brevity we write simply space, i.e., assuming each time that the space $R$ is a set on which there is defined a function $\rho(X,Y)$ of pairs of elements with the properties:

\begin{enumerate}
  \item $\rho(X,Y) = 0$ if and only if $X = Y$;
  \item for any $X,Y,Z \in R$ \[ \rho(X,Y) + \rho(Y,Z) \geq \rho(Z,X). \]
\end{enumerate}

As is well known, from (1) and (2) it follows that:

\begin{enumerate}
  \item $\rho(X, Y) = \rho(Y, X)$; $\rho(X, Y) \geq 0$.
\end{enumerate}

The elements of the set are called points of the space. The function $\rho$ is called the metric of the space, and its value is the distance between the corresponding points. Relation (2) is known as the triangle inequality.

It is natural to introduce the fundamental topological concepts into a metric space: the neighborhood of a point, convergence of sequences of points, closed and open sets, the boundary of a set, continuous mappings, continuous functions and so forth. In particular, a neighborhood of a point $A$ is any set $M \subseteq R$ which contains $A$ along with all points distant less than some positive amount from $A$. If we write $X_n \to X$, or that "the point $X_n$ converges to the point $X,"$ we mean that $\rho(X_n, X) \to 0$.

We note that $\rho(X,Y)$ is a continuous function of each of its arguments, a fact which follows from the triangle inequality.

The set $M \subseteq R$ is said to be compact (in itself) if any infinite subset
THE SIMPLEST CONCEPTS AND THE OBJECT OF THE INVESTIGATION

in it contains a sequence converging to a point of the same set. A space
is locally compact if there is a compact neighborhood of each of its
points.

2. The length of a curve. If we are given a continuous mapping
$X(t)$ of the segment $a \leq t \leq b$ into the space $R$, then we say that the
curve is defined in the concrete parametrization $X(t)$. It joins the points
$X(a)$ and $X(b)$, which are the ends of the curve.

The same point $X(t)$ may correspond to different values of $t$. The
segment $a \leq t \leq b$ decomposes into connected components $k_i$, each of
which corresponds to one and the same point $X(t)$. These components
are points or closed intervals in $[a,b]$. For the components and for the
parameter $t$ itself, it is meaningful to say that the order of succession
on $[a,b]$ is preserved. Two parametrizations $X(t), Y(s)$ ($a \leq t \leq b, c \leq s \leq d$)
are by definition equivalent if between the components $k_i$ and $k_j$ it is
possible to establish a strictly monotone 1-1 relation $\phi$ under which

$$X(k_i) = Y(\phi(k_j)).$$

Any curve is a class of equivalent parametrizations.

Any curve is called simple if the preimage of each its points $X(t)$
consists of one connected component $k(t)$. The concepts of the ends of a
curve and of the simplicity of a curve do not depend on the choice of
parametrization.

Suppose that $L$ is a curve in $R$ and $X(t)$ ($a \leq t \leq b$), is its parame-
trization. We divide up $[a,b]$ by the points $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and
form the sum $\sum_{i=1}^n \rho(X_{i-1}, X_i)$ of the successive distances between the points
$X_i = X(t_i)$. The length of the curve is the least upper bound of such sums
under all possible subdivisions:

$$s(L) = \sup \sum_{i=1}^n \rho(X_{i-1}, X_i).$$

We shall list the most important properties of length. For the proof
see for example [5, § 1, Chapter II].

1. Length is additive. If the curve $L$ is decomposed into two arcs $L_1$
and $L_2$, then

$$s(L_1 + L_2) = s(L_1) + s(L_2).$$

2. If the curve $L = X(t)$ ($a \leq t \leq b$) is rectifiable, then the length of its
arc from $X(a)$ to $X(t)$ is a continuous function of $t$. 
3. A rectifiable curve admits a parametrization with arc length as a parameter.

4. Suppose that the curves $L_n$ converge to the curve $L$, i.e., in some collection of parametric representations $X_n(t)$, $X(t)$, with the same interval $a \leq t \leq b$ of variation of the parameter, $X_n(t) \to X(t)$ uniformly in $t$. Then

$$s(L) \leq \lim_{n \to \infty} \inf s(L_n).$$

In particular, if the $s(L_n)$ are bounded uniformly, then $L$ is rectifiable.

5. If $\{L\}$ is a collection of curves whose lengths are uniformly bounded and which lie in a compact part of $\mathbb{R}$, then there exists a convergent sequence of curves in $\{L\}$.

In what follows the length of a curve $L$, measured in the metric $\rho$, will be denoted by $s^\rho(L)$.

3. **Intrinsic metric.** Suppose that $M$ is a set in the space $\mathbb{R}$. We shall call $M$ metrically connected if any two points $X, Y \in M$ may be joined by a rectifiable curve $\overline{XY}$ lying entirely in $M$. To pairs of elements of such a set one may assign the function

$$\rho_M(X, Y) = \inf_{\overline{XY} \in M} s^\rho(\overline{XY}).$$

It is not hard to verify that $\rho_M$ is a metric. It is natural to call it the metric induced by the choice of $M$ in $\mathbb{R}$, or the intrinsic metric of the set $M \subset \mathbb{R}$.

The concept of the ordinary intrinsic metric of a surface is a particularly important case of this last definition. In this case $\mathbb{R}$ is a Euclidean space and $M$ a surface. Any metrically connected surface $M$ has an intrinsic metric. This metric turns the surface $M$ into a metric space which in itself, independently of the imbedding of the surface in the enveloping space, is the object of study of the intrinsic geometry of the surface.

The new metric $\rho_M$ makes it possible to measure the length $s_{\rho_M}(L)$ of the curve $L$ lying in the set $M$. It is not hard to show that such a measure does not lead to any new result. It turns out that

$$s_{\rho_M}(L) = s_{\rho}(L).$$

This makes it possible to rewrite (6) in the form

$$\rho_M(X, Y) = \inf_{\overline{XY} \in M} s_{\rho_M}(\overline{XY}).$$
Here the question is already that of the properties of the metric $\rho_M$ itself; this time the outer space may simply not exist.

We shall agree more generally to call any metric $\rho$ intrinsic if the distance between any two points is equal to the lower bound of the length of curves joining these points:

$$\rho(X,Y) = \inf_{XY} s_p(\overline{XY}).$$

In particular, every metrically connected surface, with respect to its intrinsic metric, is a space with an intrinsic metric in the above sense. It is easy to give an example of a nonintrinsic metric $\rho^*$. For example, on the real line we may set

$$\rho^*(x_1, x_2) = \min (|x_1 - x_2|, 1).$$

The shortest arc $XY$ is the shortest of the rectifiable curves joining the points $X$ and $Y$. In order that a curve in a space with an intrinsic metric should be a shortest arc, it is necessary and sufficient that its length should be equal to the distance between its endpoints. From assertions 4 and 5 of subsection 2 it follows that in a locally compact space $R$ with an intrinsic metric there is a neighborhood $V$ of each point such that any two points in it can be joined by at least one shortest arc. It is not difficult to choose the neighborhood $V$ so that these shortest arcs do not leave the limits of a given neighborhood $U$ of the point $A$.

Along a shortest arc in a space with an intrinsic metric the length of the arc coincides with the distance between points. A curve on which each point has a neighborhood for which the segment of the curve in it is a shortest arc is called a geodesic.

2. Two-dimensional manifolds of bounded curvature.

4. Concept of an angle. The most essential feature of the intrinsic geometry of a regular surface is the curvature of the surface. We have in mind the complete Gaussian curvature (i.e., the integral of the Gaussian curvature as a function of a region on the surface). For a geodesic triangle $T$ this curvature is equal to the excess $\delta(T) = \alpha + \beta + \gamma - \pi$, where $\alpha$, $\beta$, and $\gamma$ are the angles of the triangle $T$. This well-known fact leads to the idea that one might arrive at the extension of the concept of curvature to general surfaces by starting with the concept of angle. Moreover, the concept of the angle between curves is itself the simplest concept of intrinsic geometry, next to the
concepts of distance and length.

Suppose that $L = X(t), M = Y(s)$ $(0 \leq t \leq 1, 0 \leq s \leq 1)$ are two curves in the space $R$ with the common origin $X(0) = Y(0) = O$. From $L$ and $M$ take two distinct points $X \in L, Y \in M$ and on the plane construct the triangle $T = O'X'Y'$ with the sides

$$O'X' = \rho(O, X); \quad O'Y' = \rho(O, Y); \quad X'Y' = \rho(X, Y).$$

Such a triangle exists, since the indicated distances satisfy the triangle inequality. Suppose that $\gamma(X, Y)$ is the angle in $T_0$ at the vertex $O'$. This angle also exists, since $X$ and $Y$ do not coincide with $O$.

The upper angle between the curves $L$ and $M$ at $O$ is the upper limit

$$\bar{\alpha} = \lim sup_{X, Y \to O} \gamma(X, Y). \tag{10}$$

Since $0 \leq \gamma(X, Y) \leq \pi$, the upper limit (10) necessarily exists, with $0 \leq \bar{\alpha} \leq \pi$. By $X, Y \to O$ in (10) we have in mind points $X(t), Y(t)$ distinct from $O$, which tend to $O = X(0) = Y(0)$ in the sense of the values of the parameter on $L$ and $M$.

If the limit

$$\alpha = \lim_{X, Y \to O} \gamma(X, Y) \tag{11}$$

exists, then its value is called the angle between $L$ and $M$.

5. Simple triangles. A triangle in the space $R$ is a figure consisting of three distinct points (the vertices of the triangle) and three shortest arcs joining these points (sides of the triangle).

Suppose that in the space $R$ we have selected a region $G$ open in $R$ which is homeomorphic to an open disc on the plane. Suppose that the triangle $T$ lies in this region and that its sides form a simple closed

![Figure 1](image-url)
contour. Then they bound a region in $G$. We shall attach it to $T$ and say that $T$ is a triangle homeomorphic to a disc. Here $T$ appears as a set (in the selected region $G$) with distinguished points, namely the vertices of the triangle.

Suppose that $T$ is a triangle homeomorphic to the disc, distant from the boundary of $G$ by more than one fourth of its perimeter. Then every curve joining two points on the boundary of $T$ and otherwise lying outside $T$ will, provided it is shorter than half the perimeter of $T$, lie in the region $G$ and therefore include from outside a definite piece of the boundary of $T$ (Figure 1). We shall say that such a triangle $T$ is convex relative to the boundary if no two points of its contour can be joined by a curve, lying outside $T$, which is shorter than the part of the boundary joining the points (where we mean, of course, that part of the boundary which is included by the curve).\(^1\)

A simple triangle in the region $G$ is a triangle homeomorphic to the disc and convex relative to the boundary.

Two simple triangles are said to be nonoverlapping if they do not have common interior points.

6. Fundamental definition. A space $R$ satisfying the following three requirements will be called a two-dimensional manifold of bounded curvature.

1. $R$ is a metric space with an intrinsic metric.

2. Each point has a neighborhood in $R$ homeomorphic to the disc.

3. To each point in $R$ there corresponds a neighborhood $G$ homeomorphic to the open disc, within the limits of which, for any finite system \{$T$\} of pairwise nonoverlapping simple triangles, the sum of the excesses \[\delta(T) = \bar{\alpha} + \bar{\beta} + \bar{\gamma} - \pi\] computed at the upper angles $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ of these triangles is bounded above by a number depending only on the given neighborhood $G$.

\[
\sum_{T \in \{T\}} \delta(T) \leq C(G) < \infty.
\]

Such a space is the fundamental object of investigation in the present paper.

Remark. (1) The requirement of an intrinsic character for the metric is necessary, if we wish to include the intrinsic geometry of surfaces.

\(^1\) In § 1 of Chapter III the concept of boundary convexity is extended to certain other types of figures and triangles.
This requirement excludes from consideration only metrically disconnected surfaces; for example a cylinder with a directrix in the form of a curve not rectifiable on any part of it.

(2) The requirement of the existence for each point of a neighborhood homeomorphic to the disc is also natural if we wish to be close to the intrinsic geometry of surfaces. The distinctive feature of this requirement consists of the fact that it is laid not on the pieces of the surfaces as a set in the enveloping Euclidean space but on the topological structure in the sense of the intrinsic geometry itself. This also excludes from consideration certain surfaces, for example a cone with a directrix in the form of a curve which is not rectifiable on any part of it. (This surface is metrically connected. But in its intrinsic geometry it is isometric to a continuum of isolated segments with a unique common endpoint.)

(3) From the first requirement of the definition given above it follows that the space $R$ is connected, and from the second requirement it follows that it is locally compact. From the local compactness the connectedness and the metrizability of the topological space, it follows from the theorems of P. S. Uryson and P. S. Aleksandrov [77] that the space $R$ has a countable basis. Thus in view of the first two requirements $R$ is a two-dimensional manifold.

(4) The most essential requirement is the third restriction, which we call the condition of boundedness of the curvature. If we wish to preserve concepts connected with curvature it is natural to impose a requirement of some such form. In the formulation adopted, requirement (3) is imposed in the weakest possible form. It contains only a one-sided estimate of the excesses; the excesses are considered in the sense of the always-existing upper angles; the requirement is imposed for triangles of the simplest possible form, and only locally.

(5) As we shall see in what follows, the requirements imply very comprehensive properties of the spaces. In particular, in Chapter IV we shall verify the fact that bounds of the type of inequality (12) are true in a much wider sense: for the absolute values of the excesses; for a wider class of triangles; for excesses measured with variously defined angles; in any region $G$ with a compact closure.

3. Approximation by polyhedral metrics.

7. The polyhedral metric. A special case of a two-dimensional manifold of bounded curvature is a space with a polyhedral metric. This
is a space in which each point has a neighborhood isometric to the lateral surface of a cone. An entire space of such a type, if it is closed, and a selected portion of such a space, bounded by simple closed polygons, can be defined by evolutes of plane triangles, i.e., by complexes of triangles with identified sides, the identification being carried out with preservation of the correspondence between pieces of sides of equal length. (For details on the polyhedral metric see [5, pp. 23-27].)

In a polyhedral metric there exists an angle between shortest arcs; the angle $\bar{\alpha}$ of a sector and the complete angle $\theta$ around a point have an evident meaning, and also the curvature at a point as the difference $2\pi - \theta$. Points with curvature different from zero are called vertices of the metric or essential vertices in distinction from vertices of the evolute, among which there may be nonessential vertices (with the angle $\bar{\alpha} = 2\pi$). The curvature $\omega$ of any set is defined as the sum of the curvatures of the vertices belonging to that set. The positive and negative parts $\omega^+$, $\omega^-$ of the curvature are defined as follows: the first as the sum of the curvatures for vertices with positive curvatures; the second as the absolute value of the sum of the negative curvatures. We always have $\omega = \omega^+ - \omega^-$. Finally, the absolute curvature is $\Omega = \omega^+ + \omega^-$. In a polyhedral metric for a simple polygonal line it is natural to introduce the concept of one-sided (right or left) rotation at each vertex as the difference $\pi - \bar{\alpha}$, where $\bar{\alpha}$ is the angle of the corresponding sector between the links of the line. Among the vertices of the polygonal line are included all the essential vertices of the metric through which the line passes. The rotation $\tau$ of the entire line is defined as the sum of the rotations at the vertices. Its positive and negative parts are as follows: $\tau^+$ is the sum of the positive rotations at the vertices and $\tau^-$ the absolute value of the sum of the negative rotations. Always $\tau = \tau^+ - \tau^-$. A shortest arc in a polyhedral metric evidently can not have a positive rotation on either side. Thus it can pass only through vertices with negative curvature.

For a simple closed polygon bounding a region $G$ homeomorphic to the open disc, application of Euler's theorem immediately leads to the relation

$$\tau + \omega = 2\pi,$$

where $\omega$ is the curvature of the region $G$ and $\tau$ the rotation of the boundary on the side of the region itself. This is the Gauss-Bonnet
Theorem for a polyhedron.

The concepts listed here for the polyhedral metric may now be introduced for arbitrary two-dimensional manifolds of bounded curvature. In the case of a polyhedral metric their new meaning coincides with the definitions just given.

8. Convergence of metrics. Theorems on approximation. Suppose that a sequence of metrics $\rho_n$ is given on one and the same set $M$. If uniformly for all points $X,Y \in M$

$$\lim_{n \to \infty} \rho_n(X,Y) = \rho(X,Y)$$

and if the limiting function $\rho(X,Y)$ is also a metric, then we say that the metric $\rho_n$ converge to the metric $\rho$. If we speak of the convergence of the metric spaces $R_n$ to the space $R$, we suppose fixed some 1-1 mapping $\phi_n$ of each space $R_n$ onto $R$ and we refer to the uniform convergence mentioned above of the metric $\rho_n$ on $R$ as carried over by the mappings $\phi_n$ to the metric $\rho$ of the space $R$.

In Chapter III we prove that in a two-dimensional manifold of bounded curvature each point has a neighborhood in the form of a convex polygon homeomorphic to the disc, within which $\rho$ admits uniform approximation by polyhedral metrics $\rho_n$, the absolute curvature of the latter being bounded uniformly.

This theorem opens up the possibility of investigating our spaces by approximation by polyhedral metrics. One of the first results in this direction is the proof of the existence of an angle between arbitrary shortest arcs (subsection 6 of Chapter IV).

In Chapter IV it is proved that every two-dimensional metrized manifold, whose metric admits locally uniform approximation by polyhedral metrics with positive curvatures uniformly bounded, certainly is a two-dimensional manifold of bounded curvature. This result together with the preceding remarks, gives an exhaustive characterization of our spaces as the closure of the class of spaces with polyhedral metrics. Thus we already have the possibility, mentioned in the Introduction, of uniform approximation by means of two-dimensional Riemannian spaces with uniformly bounded absolute integral curvatures.

4. Quantitative characteristics of figures. In this section we shall only mention the basic concepts. Their precise definitions and properties will be given below in the appropriate chapters of this paper.
9. **The curvature of sets.** The existence of an angle between shortest arcs makes it possible to define the excess $\delta(T)$ of a triangle, i.e., the difference between the sum of its angles and $\pi$. The function $\delta(T)$ is basic for the introduction of four functions of a set, completely additive on the ring of Borel sets (§ 2 of Chapter V). These are the curvature $\omega$, its positive and negative parts $\omega^+$ and $\omega^-$, and the absolute curvature $\Omega$. Here $\omega = \omega^+ - \omega^-$, $\Omega = \omega^+ + \omega^-$. In the case of Riemannian metrics, these functions coincide with the corresponding integral curvatures, and in the case of polyhedral metrics with the definitions of subsection 7. As in the polyhedral metric, the curvature of a one-point set coincides with the quantity $2\pi - \theta$, where $\theta$ is the complete angle of the sector around the point in question. For a triangle $T$ its curvature $\omega(T)$ generally speaking does not coincide with $\delta(T)$.

10. **Rotation of a curve.** The possibility of distinguishing the two sides of a simple arc lying in a two-dimensional manifold makes it possible to consider the approximation of the simple curve $L$ by polygonal lines $L_n$ converging to $L$ from the right and from the left (subsection 2 of Chapter VI). The measurement of sectors at the vertices of the approximating polygonal lines $L_n$ makes it possible to define the left and right rotations for a wide class of curves; these are $\tau_r$ and $\tau_l$ (subsection 3 of Chapter VI). In the case of regular curves in Riemannian spaces the left and right rotations differ only in sign and coincide with the integral geodesic curvature. More generally, however, they may be essentially different.

In the case of the so-called curves with bounded variation of the rotation (subsection 1 of Chapter IX) the usual methods of measure theory may be used to pass from the rotation of open arcs of a curve to the definition of set-functions $\tau, \tau^+, \tau^-, \sigma$ on the curve. All of these functions, the one-sided rotation, its positive and negative parts and its variation, are completely additive on Borel sets on the curve. Here $\tau = \tau^+ - \tau^-, \sigma = \tau^+ + \tau^-$. The rotation of a shortest arc on any section always turns out to be nonpositive.

11. **Rotation and curvature.** For a simple open arc that has a rotation, the sum of its right and left rotations coincides with the curvature of the arc as a point set (subsection 4 of Chapter VI). Thus the curvature distributed along the curve splits, so to speak, into two parts: the right and left rotations of the curve. When this fact is taken
into account, it is possible to follow exactly the connection between the excess and the curvature of a triangle. The excess $\delta(T)$ of a triangle computed at the angles of sectors is equal to the curvature of the triangle excluding the curvature of the vertices and that part of the curvature of the sides which is made up by the exterior rotation on the sides of the triangle.

For closed curves the Gauss-Bonnet theorem holds: if the simple curve $L$ bounds an open region homeomorphic to the disc, then $\omega(G) + \tau(L) = 2\pi$, where $\tau(L)$ is the rotation on the side of the region $G$.

12. Area. To each triangle $T$ one may put in correspondence the area $\phi$ of the plane triangle having sides with the same length. Making use of this function $\phi(T)$ in Chapter VIII we introduce the concept of the area of a figure in the spaces in question. The area is completely additive on Borel sets. The definition of the area by using the function $\phi(T)$ is similar to the introduction of curvature by means of the function $\sec(T)$. We note that the area of a triangle $T$ does not coincide with $\phi(T)$.

13. Comparison with a plane triangle. Further on we repeatedly make comparisons of a triangle with a plane triangle with sides of the same length. Such a comparison, as we have already seen, is used in the definition of the concepts of angle and area. Later on we frequently use a quantitative estimate of the difference of the angles of the original and plane triangles. This estimate is gradually improved in the course of the work. (Its most definitive form is in subsection 15 of Chapter VI and in subsection 15 of Chapter VII). An estimate of the difference of the areas of the original and plane triangles in terms of the curvature and diameter of the original triangle is established in subsection 3 of Chapter VIII. These estimates may be employed for the solution of many special questions of the intrinsic geometry of surfaces.

14. Converging and limiting figures. In Chapter VII we study various types of convergence of spaces and figures in them. We show that under uniform convergence of the metrics of two-dimensional manifolds of bounded curvature to an analogous manifold the areas of corresponding figures weakly converge. The areas of polygons converge. The curvatures weakly converge as set-functions. The positive and negative parts $\omega^+_n$, $\omega^-_n$ may not converge weakly. But every two-dimensional manifold of bounded curvature can be approximated by polyhedral metrics in such a way that the $\omega^+_n$, $\omega^-_n$ of the converging metrics converge weakly to the $\omega^+$, $\omega^-$ for the limiting metric.
In the same chapter we consider the connection at the angles between converging pairs of shortest arcs and the limiting pair of shorest arcs.

In Chapter IX we consider in detail the convergence of curves, the connection between the rotations of converging curves and the limiting curve, the cases of convergence of lengths of converging curves. The set of curves with bounded variation of rotation studied in Chapter IX may be regarded as a special kind of closure of the class of simple polygonal lines on the surface, in the same way as two-dimensional manifolds of bounded curvature form the closure of the class of polyhedral metrics. The case when the limiting curve contains a singular point with complete angle $\theta = 0$ constitutes an exception.

5. Cutting and pasting.

15. Polygons. In the spaces we are considering, polygons are compact connected sets bounded by a finite number of simple closed polygonal lines. In particular, if the space in the large is a closed manifold, then the entire space is regarded as a polygon. The definition just adopted excludes figures which could be called infinite polygons, open polygons, disconnected polygons, polygons with multiple boundary points, and so forth. In all cases of digression from the definition made here we shall make the appropriate stipulations.

Suppose that a shortest arc $AB$ is drawn from the point $A$ on the boundary of a polygon $P$, and that with the exception of the point $A$ this arc lies inside $P$. We isolate $P$ from the enveloping manifold and “cut” $P$ along the curve $AB$. Let us explain this more precisely. Each point $X$ of $AB$, except the endpoint $B$, has a neighborhood homeomorphic to a disc, which is cut by the curve $AB$ into two half-neighborhoods. Consider all the points $X \in [AB)$ in two copies of the disc. The neighborhoods of one of them are the left half-neighborhoods of the original point $X$, and the neighborhoods of the other are the right half-neighborhoods of $X$. Consequently, this set of points of $P$ with doubling of the points of the curve $[AB)$ becomes a topological space. We shall define in it an intrinsic metric, considering the distance $\rho(X,Y)$ to be the greatest lower bound of the lengths of curves joining $X$ and $Y$ which lie in the “polygon with a cut” just constructed. (Here curves lying in $P$ and intersecting the cut $AB$ are excluded.) The metric space just constructed is called a polygon with a cut. Analogously we may construct polygons with several cuts, cuts along polygonal lines or other curves, with cuts
along curves that do not abut the boundary of \( P \).

16. **Theorems on pasting.** A construction in a certain sense inverse to the cutting just described is the **pasting** of two-dimensional manifolds with an edge. Suppose that we have a finite set of two-dimensional manifolds with edges \( \mathcal{R}_j \), with interior metrics \( \rho_j \). We admit also manifolds with incomplete edges. For example, we may consider two semidiscs, in each of which the points on the circumference are not counted, but all the points of the diameter with the exception of its endpoints are counted. But we assume that the points of the edge of each manifold \( \mathcal{R}_j \) belonging to that manifold form a finite number of connected components which are simple closed curves or open arcs.

We shall divide up the boundary curves by several points, the "vertices," into a finite number of sections, the "ribs," and we map topologically various ribs taken along with the manifolds \( \mathcal{R}_j \) pairwise onto one another. We identify the points of two ribs corresponding to one another under the homeomorphism. Each "pasting" of ribs is accompanied by an identification of the corresponding vertices, namely the ends of these ribs. We suppose that no rib is sewed together with more than one other rib and that there is no identification of vertices other than that called for by the pasting together of the ribs.

The point set \( \mathcal{R}^M \) thus obtained becomes a topological space \( \mathcal{R}^T \) in the natural way. It suffices for a neighborhood of each point \( A \) not involved in the identifications to take its previous neighborhood in \( \mathcal{R} \supseteq A \), and as a neighborhood of each point \( B \) obtained by identification to take the collection of points obtained by joining the collection of neighborhoods of the point \( B \) in all the \( \mathcal{R}_j \) to which the point \( B \) belonged before the pasting.

By hypothesis the \( \mathcal{R}_j \) have intrinsic metrics and are therefore connected. We suppose that under the pasting one may pass from any \( \mathcal{R}_{j_i} \) to any \( \mathcal{R}_{j_i} \) through manifolds \( \mathcal{R}_j \) that adjoin one another under the pasting. Then for any two points \( A, B \in \mathcal{R}^T \) we can construct a chain \( Z(A,B) \) which is a finite collection of points \( X_i \in \mathcal{R}^T \), the first of which is \( A \) and the last is \( B \), and each two successive \( X_i, X_{i+1} \) belong to one and the same original manifold \( \mathcal{R}_{j_i} \).

Define a function \( \rho \) in \( \mathcal{R}^T \) by putting

\[
\rho(A,B) = \inf_{Z(A,B)} \left[ \sum_i \rho_{j_i}(X_i,X_{i+1}) \right].
\]

It is not hard to verify that \( \rho(A,B) \) is an intrinsic metric. We shall
call the space $R$ with the metric $\rho$ thus obtained a space \textit{pasted up} from the manifolds $\bar{R}_j$.

If all the sections of the boundary curves of the manifolds $\bar{R}_j$ undergo pasting, then $R$ becomes a two-dimensional manifold. In the contrary case it becomes a manifold with an edge or a manifold with an incomplete edge.

The simplest case of pasting consists of defining a polyhedral metric by evolutes of plane polygons.

In Chapters VI and IX we prove the following two theorems on pasting: 1) if the two-dimensional manifold $R$ is pasted up from manifolds $\bar{R}_j$ with edges, where the $\bar{R}_j$ are manifolds with intrinsic metrics excised from two-dimensional manifolds of bounded curvature, and if under the pasting the identification of boundaries took place with sections of equal length corresponding, then the space $R$ itself is a two-dimensional manifold of bounded curvature; 2) if the two-dimensional manifold $R$ is pasted up from manifolds $\bar{R}_j$ with boundaries, and if the $\bar{R}_j$ with their intrinsic metrics are excised from two-dimensional manifolds of bounded curvature, where the $\bar{R}_j$ were bounded by a finite number of simple curves with bounded variation of the rotation, and if the identification of the boundaries took place with correspondence of sections of equal length, then the space $R$ itself is also a two-dimensional manifold of bounded curvature.

The second theorem contains the first as a special case. In both cases each curve of the pasting conserves its rotation on the side of the corresponding $\bar{R}_j$, and at the vertices of the pasting the angles of the pasted sectors are preserved.

The operations of cutting and pasting are a convenient method of investigation. They make it possible, while preserving many properties to rebuild the enveloping space.

6. Further investigations. In this section we shall list further results from the geometry of two-dimensional manifolds of bounded curvature which do not lie within the scope of the present monograph. A more detailed survey will be found in [17, § § 5–9].

17. Linear element. As Ju. G. Rešetnjak [61], [64], showed, on a piece of any two-dimensional manifold of bounded curvature homeomorphic to the disc with $\omega^+ < 2\pi$ it is possible to introduce an isometric coordinate net in which a metric linear element of the space will have
the form $ds^2 = \lambda(u,v) (du^2 + dv^2)$, where $\ln \lambda(u,v)$ is the difference of two subharmonic functions. Moreover, the possibility of introducing a net with those properties is not only a necessary but a sufficient test for a two-dimensional manifold of bounded curvature. Later this result was proved again by A. Huber [79]. Ju. G. Rešetnjak also cleared up the degree of arbitrariness with which the indicated net can be introduced. In the second of the papers just mentioned the author used the linear element for the study of two-dimensional manifolds of bounded curvature, and in particular for the analytic description of curves with bounded variation of rotation, in connection with the properties of preimages of these curves on the coordinate plane. These results establish the relationship of the theory in question with classical differential geometry and the theory of functions, and also give yet another general method of investigation. Apparently, similar results hold also for other types of nets. For Čebyšev nets a part of the investigation has already been carried out by I. Ja. Bakel’man.

The question as to what minimal properties of curvature guarantee the possibility of prescribing a metric by a regular linear element is an interesting one. In the case of convex surfaces, it was proved by A. D. Aleksandrov [5, Chapter XI] that for this purpose it suffices that there exist at each point a specific Gaussian curvature as the limit of the ratio of the curvature of an arbitrary region to the area of that region as the region contracts to a point.

18. Special questions of intrinsic geometry. The methods developed by A. D. Aleksandrov make it possible to pose and resolve many special questions of intrinsic geometry for two-dimensional manifolds of bounded curvature. We shall mention a number of them.

Extremal problems. The question of the maximum area of a region homeomorphic to the disc with a bounded positive part of the curvature $\omega^+ \leq a < 2\pi$ and a bounded perimeter or diameter is a characteristic example of a problem in the calculus of variations, the solution of which is achieved on a nonregular surface. A. D. Aleksandrov solved this problem by approximating by polyhedra and by applying the method of cutting and pasting [2], [5], [17]. The papers of S. M. Lozinskiï, [52], Ju. G. Resetnjak [65], and A. Huber [78], touch on related questions. Here one must mention also the paper [68] of G. I. Rusiešvili. Moreover, A. D. Aleksandrov in [7], [17], [18] and V. V. Strel’cov in [18], [71], [72] obtained interesting estimates of the length of a curve with a
bounded variation of rotation, lying in a region homeomorphic to the disc with $\omega^+ < 2\pi$ and having bounded perimeter or diameter. The question of such estimates was already present in the paper of Cohn-Vossen [50].

**Geodesics and quasigeodesics.** For nonregular convex surfaces A. D. Aleksandrov introduced the concept of a quasigeodesic curve [5] as a curve with a nonnegative right and left rotation on each section. A. V. Pogorelov investigated these curves in detail [54]. They turn out to be the natural closure of the class of geodesics. In order that a curve on a convex surface should be quasigeodesic it is necessary and sufficient that it be the limit of geodesic curves which lie on convex surfaces converging to the given surface. A. V. Pogorelov established the existence of three simple closed quasigeodesics on any closed convex surface, a generalization of a theorem of L. A. Ljusternik and L. G. Shnirel'man. In a two-dimensional manifold of bounded curvature a quasigeodesic is defined as a curve for which the sum of the variations of the right and left rotation coincides with the absolute curvature. In [9] and [10] A. D. Aleksandrov also investigated quasigeodesics by the methods of intrinsic geometry, and generalized the results of Cohn-Vossen on the behavior in the large of geodesic curves (A. D. Aleksandrov [7], V. V. Strel'cov [73], Ju. F. Borisov [30]).

**Circumference and disc.** On a convex surface a circumference with a fixed center is a simple closed curve only for small values of the radius. The length of the circumference and the area of the disc as a function of the radius are functions of bounded variation: they admit simple estimates in terms of the value of the radius and the curvature of the disc. The properties of the circumference make it possible to establish an almost isometric mapping of the neighborhood of a point onto the tangent cone. These results, proved by V. A. Zalgaller [44] for convex surfaces, admit a natural generalization for two-dimensional manifolds of bounded curvature, if the center of the circumference lies at a point with a complete angle different from zero.

**Manifolds with an edge.** Ju. F. Borisov investigated in detail the singularities of metrized two-dimensional manifolds with an edge [30],

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2 Half the difference between the sum of the variation of the right and left curvature of a curve and its absolute curvature is called the proper rotation or twist of the curve. Thus quasigeodesics have a zero proper rotation.
Moreover, he investigated the structure of a semineighborhood of a curve and the variation of the length of a curve under one-sided displacement of the curve on a two-dimensional manifold of bounded curvature [32].

**Manifolds with bounded specific curvature.** It is possible to distinguish a class of two-dimensional manifolds by the property of being intermediate between spheres and pseudospheres of various radii. These are spaces in which the ratio of the curvature to the area for any region lies within the limits $K_1 \leq \omega/s \leq K_2$. One may obtain various estimates concerned with the comparison between figures in these manifolds and the corresponding figures on surfaces of constant curvature $K_1$ and $K_2$. See A. D. Aleksandrov [2], [3], [5, Chapter X], [13], [15], [17, §7], V. V. Strel’cov [74], [75], and G. I. Rusiešvili [68].

19. **Surfaces in a space.** The most studied among the nonregular surfaces are the general convex surfaces. For these the negative part of the intrinsic curvature $\omega^- = 0$. In general, every two-dimensional manifold of bounded curvature with $\omega^- = 0$ is locally isometric to a piece of a convex surface (see A. D. Aleksandrov [5], [6]). For convex surfaces the close connection of intrinsic and extrinsic geometric properties is preserved. First of all we have the generalized theorem of Gauss: the area of a spherical representation coincides with the intrinsic curvature of the same set (theorem of A. D. Aleksandrov [5]). Moreover, every shortest arc on a surface has at each of its points a right and left semitangent. In other words, it has directions in the space (theorem of I. M. Liberman [51]). Moreover, every curve with a bounded variation of rotation on a convex surface has a finite rotation in the space (theorem of V. A. Zalgaller [45]). The corresponding results are far from always holding for nonconvex surfaces. It suffices to note that the theorem of Liberman may fail to hold even for smooth surfaces isometric to the plane.

A series of papers have been devoted to the singling out of a class of nonconvex surfaces which with respect to their intrinsic geometry are two-dimensional manifolds of bounded curvature and to establishing connections between the extrinsic and intrinsic geometry of such surfaces.

A. D. Aleksandrov [8], [11] considered surfaces “representable as the difference of convex surfaces,” i.e. admitting the representation $z = f(x,y)$, where $f$ is the difference of two convex functions. The paper of Ju. G. Rešetnjak on “generalized convex” surfaces [62] (i.e., surfaces
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tangent at each point on a definite side to a sphere of fixed radius)\(^3\) deals
with basically the same class of surfaces. All of these surfaces are two-
dimensional manifolds of bounded curvature, and the generalized Gauss
theorem and the Liberman theorem hold for them.

I. Ja. Bake' man [21]–[25] considered smooth surfaces "with general-
ized second derivatives." These are surfaces locally admitting the parame-
trization \(r(u, v)\), where the vector-function \(r\) has continuous first
derivatives and square-summable generalized second derivatives. For
these surfaces the fundamental relations of differential geometry are
preserved in integral form. A. V. Pogorelov [57]–[59] investigated
smooth surfaces of "bounded extrinsic curvature," in other words,
surfaces with finite area for the spherical representation. For all the
surfaces mentioned in this part it has been proved that they are two-
dimensional manifolds of bounded curvature and that the theorem
of Gauss holds for them. The properties of curves on these surfaces
have still not been studied.

The somewhat more general class of surfaces with bounded extrinsic
curvature [25] considered by I. Ja. Bake' man, include the surfaces
enumerated above. For this class only one theorem has been proved:
with respect to its intrinsic geometry, every surface in this class un-
questionably is a two-dimensional manifold of bounded curvature.

For metrics of positive curvature there have been established the
theorems of A. D. Aleksandrov [5] on their realization on convex
surfaces, and also the theorems of A. V. Pogorelov on the uniqueness
of the realization for a closed surface (or under prescribed boundary
conditions) [56], and on the regularity of the realization for sufficiently
regular metrics [55]. The situation is quite different in the case of
general metrized manifolds. From the work of John Nash [53] and N.
Kuiper [49] it follows that various realizations of Riemannian spaces
\(R^2\) are possible in the form of smooth surfaces in \(E^3\). The realization
of nonconvex polyhedral evolutes was treated in the papers of V. A.
Zalgaller [48] and of Ju. D. Burago and V. A. Zalgaller [39]. The
existence of a surface realizing the metric of any closed oriented two-
dimensional manifold of bounded curvature was established by Ju. D.
Burago [38]. However, there remains the important question of dis-
tinguishing a sufficiently general class of surfaces in which pieces of

\(^3\) For a more precise definition see [62].
any two-dimensional manifold of bounded curvature can be realized and for which results of the type of the Gauss theorem and the Liberman theorem would be valid.

20. Generalizations. Multidimensional polyhedral evolutes were used by Ju. A. Volkov [41]. The Gauss-Bonnet theorem for such evolutes was established by I. A. Brin [37]. But the general definition of two-dimensional manifolds of bounded curvature did not lead to any multidimensional analogue.

Independently of the number of dimensions, A. D. Aleksandrov considered metric spaces “with curvature less than $K$” [13], [15]–[17]. These are spaces in which the excess in any triangle with respect to the upper angles is not larger than the triangles with sides of the same length on a surface of constant Gaussian curvature $K$. Related questions were considered in the papers of V. A. Toponogov [76] and Ju. G. Rešetnjak [65], [67].

Nonregular smooth surfaces, possibly not satisfying the condition of bounded curvature (subsection 6 of Chapter I) but admitting the introduction of a parallel translation, were studied by Ju. F. Borisov [35], [36].

Some theorems were obtained by Ju. G. Rešetnjak [63] for metrized two-dimensional spaces with the single requirement that a tangent cone exists at each of their points.
Chapter II

Angle

1. Properties of the upper angle. As we have already said, in any metric space the upper angle between the curves \( L \) and \( M \) at their common origin \( O \) is the upper limit

\[
\bar{\alpha} = \limsup_{x, y \to 0} \gamma(X, Y),
\]

where \( X \in L, \ Y \in M; \ X, Y \not\equiv O; \ X, Y \) tend to \( O \) in the sense of the parameters on \( L \) and \( M \).

Analogously the lower angle is the lower limit

\[
\underline{\alpha} = \liminf_{x, y \to 0} \gamma(X, Y).
\]

Evidently we always have \( 0 \leq \underline{\alpha} \leq \bar{\alpha} \leq \pi \).

If \( \underline{\alpha} = \bar{\alpha} \) their common value \( \alpha \) is called simply the angle between \( L \) and \( M \).

We note that if \( L \) and \( M \) are shortest arcs and if for some \( X, Y \) the angle \( \gamma(X, Y) = \pi \), then also for all points \( X, Y \) closer to \( O \) the angle \( \gamma = \pi \) and therefore there is an angle between \( L \) and \( M \) equal to \( \pi \).

1. The triangle inequality for upper angles.

Theorem 1. If in the space \( R \) there are three curves \( L_1, L_2, \) and \( L_3 \) issuing from the point \( O \) and forming the upper angles \( \bar{\alpha}_{12}, \ \bar{\alpha}_{13}, \ \bar{\alpha}_{23}, \) then the sum of any two of these angles is not less than the third.

Proof. In view of the symmetry it suffices to establish the inequality

\[
\bar{\alpha}_{13} \leq \bar{\alpha}_{12} + \bar{\alpha}_{23}.
\]

If \( \bar{\alpha}_{13} = 0 \) this is obvious. Suppose that \( \bar{\alpha}_{13} > 0 \) and \( X_n \in L_1, \ Y_n \in L_3 \) are pairs of points converging to \( O \) for which the corresponding angles \( \gamma_{13} \to \bar{\alpha}_{13} \).

1. We suppose first that among the angles \( \gamma_{13} \) there are infinitely many distinct from \( \pi \). We keep only those pairs \( X_n, \ Y_n \) for which \( 0 \equiv \gamma_{13} \equiv \pi \) and fix attention on one of them. The plane triangle \( O'X_n'Y_n' \) with sides equal to \( OX_n, \ OY_n, \ X_nY_n \) will have one of the two forms depicted in Figure 2 (for definiteness we suppose \( OX_n \geq OY_n \)).
If we continuously move the point $Z$ from far out along the curve $L_2$ towards its origin $O$, there is a last moment when $OZ = O'X'_n$, and later a first moment when $OZ$ is equal to the altitude $O',P$ of the triangle $O'X'_nY'_n$. We shall move the point $Z$ along $L_2$ only on the section between these two positions. Moving $Z$, we can mark off on $X'_nP$ a point $Z'$ for which $O'Z' = OZ$. If the triangle $O'X'_nY'_n$ has the form depicted on Figure 2a, then the point $Z'$ gradually shifts from $X'_n$ into $Y'_n$. But if the triangle has the form depicted in Figure 2b, then the point $Z'$ on moving along the selected part of the curve $L_2$ shifts from $X'_n$ to $P$. Then by moving $Z$ in the reverse direction and putting $Z'$ off onto the segment $PY'_n$, we cause $Z'$ to shift into $Y'_n$.

In both cases we have obtained a continuous process in which $Z$ moves along $L_2$, not encountering $O$, and $Z'$ shifts from $X'_n$ to $Y'_n$. In the course of this process there is a position $Z_*$ of the point $Z$ for which the following equation is satisfied:

$$\frac{X'_nZ'}{X'_nZ' + Z'_nY'_n} = \frac{X_nZ}{X_nZ + ZY_n}.$$  

Indeed, the right side of this equation is continuous and remains included between 0 and 1, and the left side varies from 0 to 1. There must be at least one point where they are equal.

Since $X'_nZ'_n + Z'_nY'_n = X_nY_n \leq X_nZ_n + Z_nY_n$, there follow separately from (3) the inequalities $X'_nZ'_n \leq X_nZ_n$ and $Z'_nY'_n \leq Z_nY_n$. Therefore in the plane triangles with the sides $OX_n$, $OZ_n$, $X_nZ_n$ and $OZ_n$, $OY_n$, $Z_nY_n$ the angles $\gamma_{12}$ and $\gamma_{23}$ are not less than the angles $\angle X'_nO'Z'_n$ and $\angle Z'_nO'Y'_n$, which sum to the angle $\gamma_{13}$.
Thus for the points $X_n, Y_n, Z_n$ selected above we have $\gamma_{13} \leq \gamma_{12} + \gamma_{23}$, so that as $n \to \infty$ we obtain (2).

2. Consider the remaining case, where in the sequence $X_nY_n$ all the angles $\gamma_{13} = \pi$, i.e., $X_0O + OY_n = X_nY_n$.

We assert that for fixed $X_n, Y_n, \varepsilon > 0$, it is possible, by choosing $Z_n$ on $L_2$ sufficiently close to $O$, to assure that the sum of the angles $\gamma_{12} + \gamma_{23}$ should not be less than $\pi - \varepsilon$. Indeed, if this were not so, we would have for $Z_n$ close to $O$ a plane quadrilateral with a convex angle at the vertex $O'$ and a reentrant angle at the vertex $Z'_n$ (Figure 3). This is not possible in the case at hand, since

$$X'_nZ'_n + Z'_nY'_n = X_nZ_n + Z_nY_n \geq X_nY_n = X'_nO' + O'Y'_n.$$  

Thus in this case $\bar{\alpha}_{13} = \pi \leq \bar{\alpha}_{12} + \bar{\alpha}_{23}$ and Theorem 1 is completely proved.

In what follows when we speak of shortest arcs, we shall always mean only those shortest arcs whose length is equal to the distance between their endpoints. In a space with an intrinsic metric all shortest arcs have this property.

It follows from the definition of angle that two exemplars of one and the same shortest arc form at the endpoint a zero angle, since in this case $\gamma(X,Y) = 0$. Two branches of one and the same shortest arc at each of its interior points also form a definite angle equal to $\pi$, since in this case $\gamma(X,Y) = \pi$. If still another curve issues from an interior point of a shortest arc, then it forms an adjacent angle with the two branches of the shortest arc. From Theorem 1 we obtain the following.

**Theorem 2.** The sum of adjacent angles is not less than $\pi$.

**Theorem 3 (Sufficiency Test for the Additivity of Angles).** Suppose that the shortest arcs $L_1, L_2, \ldots, L_n$ issue from the point $O$, while there exist the angles $\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1,n}, \alpha_{1n}$.

Then if there exist points $X_1 \in L_1, X_n \in L_n$ which are arbitrarily close to $O$ and which may be joined by the shortest arc $X_1X_n$, which successively intersects $L_2, \ldots, L_{n-1}$ at the points $X_2, \ldots, X_{n-1}$ distinct from $O$, then

(4)  \[ \alpha_{1n} = \alpha_{12} + \alpha_{23} + \cdots + \alpha_{n-1,n}. \]

**Proof.** Construct the plane triangles with sides $OX_i, OX_{i+1}, X_iX_{i+1}$ ($i = 1, \ldots, n-1$) and successively apply them to one another (we denote points in the plane by a prime as in Figure 4). Straightening out the curve $X'_1X'_2 \cdots X'_{n-1}X'_n$, we conclude that

(4')  \[ \gamma_{1n} \geq \gamma_{12} + \gamma_{23} + \cdots + \gamma_{n-1,n}. \]
therefore in the limit
\[ \alpha_{1n} \geq \alpha_{12} + \alpha_{23} + \cdots + \alpha_{n-1,n}. \]
But from Theorem 1 we get
\[ \alpha_{1n} \leq \alpha_{12} + \alpha_{23} + \cdots + \alpha_{n-1,n}, \]
which along with the preceding gives (4).

2. Direction. We shall say that a curve issuing from a point has a direction if at the initial point it forms with itself a definite angle, in other words if it forms a definite angle with a second copy of the same curve. This angle can obviously only be zero, since in the given case \( \alpha_\cdot = 0 \). Therefore it is sufficient, and obviously necessary, for the curve to have a direction, that the upper angle \( \bar{\alpha} \) formed by the curve with itself at the initial point should be zero.

A shortest arc obviously has a definite direction at its endpoints.

We shall say that two curves have at the origin the same direction if each of them has a direction and they form a zero angle with one another. The property of curves of having the same direction is reflexive, symmetric and transitive. The last follows from the fact that if \( \bar{\alpha}_{12} = 0 \) and \( \bar{\alpha}_{23} = 0 \), then since \( \bar{\alpha}_{13} \leq \bar{\alpha}_{12} + \bar{\alpha}_{23} \) we get \( \bar{\alpha}_{13} = 0 \). Therefore the collection of curves issuing from a point and having direction decomposes into classes of curves having the same direction. By a direction itself we shall understand simply each of the classes of curves having the same direction.

The collection of directions issuing from a fixed point of a metric space in its turn forms a metric space, the space of directions, in which the angular distance between directions, i. e., between classes of curves having the same direction, is by definition the upper angle between arbitrary curves of the corresponding classes. From Theorem 1 this angular distance has all the properties of a metric.

Remarks. (1) In a Euclidean space curves with a directed semitangent at the origin have a direction in the indicated sense. Curves with the same semitangent have the same direction. For a Riemannian space the property of a curve of having a direction coincides with the same property
for its image in a tangent Euclidean space.

(2) In the case of Euclidean and Riemannian spaces there exist unique shortest arcs from a point in each direction. In the general case this is not so. On a convex surface there may be points (for example on the edge of the base of a convex circular cone) at which there are in certain directions no issuing shortest arcs. On nonconvex surfaces there may also be directions in which there issue more than one shortest arc.

(3) At each point of an \( n \)-dimensional Euclidean or Riemannian space the space of directions is isometric to an \((n-1)\)-dimensional Euclidean sphere. On a convex surface it is isometric to the collection of generators of the tangent cone, if the distance between generators is measured by the angle of the sector on the cone. In general, the space of directions can be quite arbitrary.

(4) It is easy to see that every metric space in which there are no points distant more than \( \pi \) is isometric to a space of directions at some point of another metric space. Indeed, if to each element \( x \) of the given space \( R \) we assign a unit segment \( x' \) from an abstract bundle of segments with a common origin \( O \) and define the distance between points on the segments \( x', y' \) as the distance of the corresponding points on the sides of a plane sector with an angle equal to the distance \( \rho(x, y) \) in \( R \), then we obtain a new metric space in which the space of directions at the point \( O \) is isometric to \( R \).

(5) Further, for a space which is a manifold, the space of directions may at certain points turn out to be quite peculiar. Consider, for example, the space of a circular cylinder. All the points of the base will be identified with a single point \( O \). Each pair of points \( X, Y \) of the generating set is joined by two paths: the shortest arc on the lateral surface of the cylinder and the path from \( X \) along the generator to the base and from the base (now a single point) along another generator to the point \( Y \). The smaller of the lengths of these two paths will be called the distance \( \rho(X, Y) \). In such a space there is a neighborhood of the point \( O \) homeomorphic to a disc. We can show that curves which issue from \( O \) and have at \( O \) a definite direction are those and only those curves which are tangent to one of the generators of the cylinder close to \( O \). Any pair of such curves, tangent to distinct generators, forms an upper angle \( \bar{\alpha} = \pi \). In this example the space of directions at the point \( O \) consists of a continuum of elements pairwise \( \pi \) units distant from each other.
(6) The direction was defined above in a purely intrinsic fashion, in terms of the metric of the space itself. If we are given two metrics in a space, defining the same topology, in particular if the space is imbedded in a wider metric space, as for example a surface with its intrinsic metric imbedded in the enveloping Euclidean space, then the property of a curve of having a direction, in the sense of each of the two metrics, may generally speaking not coincide. In the case of regular surfaces in the usual Euclidean space, and also for general convex surfaces, the properties of a curve of having or not having a direction in the sense of the metric of the enveloping space or of the metric of the surface itself coincide, but even the usual plane may be bent in such a way that some straight line on it turns into a curve not having a direction in the enveloping space. The possibility of such singularities leads to difficulties in the investigation of the connection of the spatial form and the intrinsic metric of general nonconvex surfaces.

(7) In the simplest two-dimensional spaces the property of a curve of having a direction and the ability to form angles with other curves are closely connected. On a plane, on a regular surface and on an arbitrary convex surface two curves form a definite angle if and only if each of them has a definite direction. The situation is different in the many-dimensional case and more general surfaces. Thus in the usual three-dimensional space the axis of a circular cone and a spiral situated on the surface of that cone form at the vertex of the cone a definite angle, equal to the angle between the generators and the axis of the cone, while one of these curves, namely the spiral, has no definite direction at the origin. In the two-dimensional case, on the surface of a cone with a complete angle larger than $2\pi$ the curves $L_1$ and $L_2$ (Figure 5) have no definite directions at the point $O$, but form pairwise an angle equal to $\pi$. In many spaces on the other hand it may happen that no angle exists, even between shortest arcs, which certainly have definite directions.

We shall return to the questions touched on here in subsection 1 of Chapter VI, especially for the case of two-dimensional manifolds of bounded curvature.
3. Characteristic stability of the upper angle.

**Lemma 1.** For the angle opposite the side $z$ of a plane triangle $T$ with sides $x,y,z$ we have

$$\cos \gamma = \frac{y - z}{x} + \varepsilon,$$

where $\varepsilon \to 0$ when $x/y \to 0$.

**Proof.** We have:

$$\cos \gamma = \frac{x^2 + y^2 - z^2}{2xy} = \frac{y - z}{x} + \frac{x^2 - (z - y)^2}{2xy}.$$

But $|z - y| \leq x$, so that

$$\frac{y - z}{x} \leq \cos \gamma \leq \frac{y - z}{x} + \frac{1}{2} \cdot \frac{x}{y},$$

which proves the lemma.

The following theorem demonstrates the stability characteristic of the upper angle between shortest arcs.

**Theorem 4.** If a curve $L$ and a shortest arc $M$ issue from the point $O$, then the upper limit of the angles does not increase if instead of $X,Y \to O$ we require that only the point $X \in L$ approach $O$, and that the point $Y$ move in any way on the shortest arc $M$. That is, in this case

$$\bar{\alpha} = \lim \sup_{X \to O} \gamma(X,Y).$$

**Proof.** If the upper limit on the right in (7) is attained on a sequence $OX_n = x_n \to 0$, $OY_n = y_n \to 0$, it must thus coincide with the upper angle $\bar{\alpha}$. Suppose that it is attained for $x_n \to 0$, $y_n \geq a > 0$.

We mark off on $M$ a variable point $Y'_n$ such that $y'_n \to 0$ and $y'_n/x_n \to 0$. Then from the triangle inequality we have:

$$Y'_nY_n \geq X_nY_n - X_nY'_n,$$

i.e., $y_n - y'_n \geq z_n - z'_n$,

so that

$$\frac{y_n - z_n}{x_n} \geq \frac{y'_n - z'_n}{x_n}.$$

From Lemma 1 this gives

$$\lim \inf \cos \gamma(X_n, Y_n) \geq \lim \inf \cos \gamma(X_n, Y'_n) \geq \lim \inf \cos \gamma(X, Y),$$

from which

$$\lim \sup \gamma(X, Y) \leq \bar{\alpha}.$$
Since the inverse inequality is obvious this result implies (7). Theorem 4 is proved.

Remark. Assertions analogous to Theorems 1, 2, and 4 do not hold for lower angles. Theorem 4 generally ceases to be valid if $M$ is not a shortest arc.

4. Upper angle and the distance along shortest arcs. Suppose that $L$ is a given shortest arc, $Y_0$ a point not on $L$, $X$ a point on $L$, $M=OY_0$ any shortest arc joining $Y_0$ to $O$ (we assume that there is at least one such shortest arc), and $\bar{\alpha}$ the upper angle between the left branch of $L$ and $M$ (Figure 6). Consider the variable point $X$ on $L$. Its position may be characterized by the coordinate $x$, measured from the origin $A$ of the shortest arc $L$. The distance $Y_0X=z$ is a function of $x$. The left upper and lower derivatives of the function $z(x)$ with respect to $x$ and corresponding to the point $O$, are the upper and lower limits of the ratio $(XY_0-OY_0)/OX$, or $(y-z)/x$, if by $x$, $y$ and $z$ we understand the distances $OX$, $OY_0$ and $XY_0$ themselves. From Lemma 1

$$
\cos \gamma - \varepsilon \leq \frac{y-z}{x} \leq \cos \gamma.
$$

For $y=\text{const}$, $x\to 0$ the ratio $x/y\to 0$, and along with it $\varepsilon\to 0$. Therefore $(\partial z/\partial x)_{LU}$ and $(\partial z/\partial x)_{LL}$ are the upper and lower limits of $\cos \gamma$, when $Y=Y_0$ and the point $X$ approaches $O$ from the left.

The lower limit of $\cos \gamma$ can only decrease if we remove the condition $Y=Y_0$ and permit $Y$ to take arbitrary positions on $OY_0$. In view of Theorem 4 this leads us to the following assertion.

Lemma 2. For $x$ corresponding to the point $X=O$,

$$
\cos \bar{\alpha} \leq \left( \frac{\partial z}{\partial x} \right)_{LL}
$$

2. Lower strong angle. The concept introduced here may seem rather artificial. But its properties are used in an essential way in what follows. In a special supplement to Chapter II at the end of the paper we discuss in more detail the possible definitions of an angle, and their various properties and interrelations.
5. **Angle in the strong sense.** Suppose that \( L \) and \( M \) are completely determined arcs of shortest curves, with \( L = \overline{OX_0} \) and \( M = \overline{OY_0} \). Consider all possible sequence of points \( X_n, Y_n \) for which the following conditions are satisfied:

a) \( X_n \in \overline{OX_0}, \ Y_n \in \overline{OY_0}; \ x_n \neq 0, \ y_n \neq 0; \ x_n \to 0 \) or \( y_n \to 0 \);

b) each pair \( X_n, Y_n \) admit a joining by a shortest arc \( \overline{X_nY_n} \) such that in the case \( x_n \to 0 \) the shortest arc \( \overline{X_nY_n} \) converges to a segment of the shortest arc \( M \), and in the case \( y_n \to 0 \) the shortest arc \( \overline{X_nY_n} \) converges to a segment of the shortest arc \( L \).

The lower limit of the angles \( \gamma(X_n, Y_n) \), taken on all possible sequences \( X_nY_n \) satisfying conditions a) and b) is called the **lower strong angle** between \( L \) and \( M \):

\[
\alpha_{(-)s} = \lim \inf \gamma(X_n, Y_n).
\]

Correspondingly the upper strong angle is defined as follows:

\[
\alpha_s = \lim \sup \gamma(X_n, Y_n).
\]

In a locally compact space conditions a) and b) are satisfied by any sequence of points \( X_n, Y_n \to 0 \) along \( L \) and \( M \). Therefore in such spaces the limits (9) and (10) surely exist and

\[
0 \leq \alpha_{(-)s} \leq \alpha_- \leq \bar{\alpha} \leq \bar{\alpha}_s \leq \pi.
\]

If \( \alpha_{(-)s} = \bar{\alpha}_s = \alpha_s \), then we say that there is an angle between a \( L \) and \( M \) in the strong sense, or a strong angle \( \alpha_s \).

**Remarks.**

1) The peculiar features of these last definitions are the extension of the freedom in the positions of the points \( X_n \) and \( Y_n \), only one of them having to converge to \( 0 \), and the requirement that there must be some connection between the shortest arcs \( \overline{X_nY_n} \) and the shortest arcs \( L \) and \( M \). The restriction b) is quite natural. If there exists a shortest arc \( L' \) distinct from \( L = \overline{OX_0} \) with the same endpoints, then without restriction b) the sequence \( X_n = X_0, \ Y_n \to 0 \) would equally characterize the pair \( L, M \) and \( L', M \). Such an example can be constructed on a convex surface.

2) Theorem 4 implies that \( \bar{\alpha}_s = \bar{\alpha} \). Hence \( \bar{\alpha}_s \) is not a new characterization of an angle.

3) The concept of a lower strong angle of interest to us here is rather unwieldy. But it enables us to obtain a number of estimates that are symmetric to the estimates obtained by using the upper angle, something not possible with the ordinary lower angle.
4) An unsatisfactory feature of the latter definitions is that they are not connected with the course of $L$ and $M$ in an arbitrarily small neighborhood of the point $O$, but with the properties (and thus with a concrete choice) of entire sections $\overline{OX}_0$, $\overline{OY}_0$ of shortest arcs. In this sense we are dealing with nonlocal characteristics of the angle between curves. Since $\alpha_s = \bar{\alpha}$, what we have just said relates only to $\alpha_{(\cdot)}$. As soon as we have succeeded in showing that between certain shortest arcs there is an angle in the strong sense, all the estimates established with the use of the lower strong angle acquire a local character.

6. Strong angle and distance along a shortest arc. Suppose that under the conditions of subsection 4 (Figure 6) there exists on $L$ and $M$ from $O$ at least one sequence of points $X_n \to O$ for which there exist shortest arcs $X_nY_0 \to OY_0$. We define the left upper strong derivative $(\partial z/\partial x)_{LUS}$ as the upper limit $\Delta z/\Delta x$ taken only on these sequences of points $X_n$.

From the connection of $\Delta z/\Delta x$ with the values of $\cos \gamma(X_n, Y_n)$ and the fact that if we drop the restriction $Y = Y_0$ the upper limit can only decrease, we arrive at the following assertion, to some extent opposite to Lemma 2.

**Lemma 3.** For $x$ corresponding to the point $X = O$,

\[
\left( \frac{\partial z}{\partial x} \right)_{LUS} \leq \cos \alpha_{(\cdot)} ,
\]

where $\alpha_{(\cdot)}$ is the lower strong angle between the shortest arcs $\overline{OY}_0$ and an arbitrarily small section of the left branch of $L$.

3. Fundamental theorems on the angles of a triangle.

7. Variation of the angle $\gamma$. Consider two shortest arcs $L$ and $M$ issuing from the point $O$, and on them the points $X$ and $Y$. Suppose as usual that $x, y$, and $z$ are the distances $OX, OY, XY$. Evidently, if $X$ is displaced along $L$ the value $y$ is preserved and $|\Delta z| \leq |\Delta x|$. We shall be interested in the variation of the angle $\gamma$.

**Lemma 4.** If in a plane triangle with sides $x, y$, and $z$ and angles $\gamma$ and $\xi$ opposite $z$ and $y$, the lengths $x$ and $z$ vary such that $|\Delta x| \leq |\Delta z|$ while preserving the length $y$, then

\[
\frac{\Delta z}{\Delta x} = \cos \xi_0 + x \frac{\Delta \gamma}{\Delta x} \sin \xi_0 + \varepsilon,
\]

where $\varepsilon \to 0$ as $\Delta x \to 0$. 
Proof. Suppose that \( OXY \) is the original and \( OXY' \) the varied triangle, as in Figure 7. We drop a perpendicular \( YM = l \), from the point \( Y \) to the side \( OY' \) or to its extension. From its base \( M \) and from the point \( Y \) we drop perpendicularly \( MP \) and \( YN \) onto \( XY' \). Then we have:

\[
OM \approx x, \quad XN \approx z, \\
NP \approx l \sin \xi_0, \\
MY' \approx \Delta x, \quad NY' \approx \Delta z, \\
PY' \approx \Delta x \cos \xi_0,
\]

where the approximate equations are true up to terms of higher orders of smallness with respect to \( |Ax| \). Therefore from \( NY' = NP + PY' \) it follows that

\[
\Delta z = \Delta x \cos \xi_0 + l \sin \xi_0 + \varepsilon, Ax.
\]

But from the triangle \( OMY \), by the law of sines,

(13)

\[
l = x \sin \Delta \gamma.
\]

Taken together with our previous results, this equality, after replacement of \( \sin \Delta \gamma \) by the equivalent infinitesimal \( \Delta \gamma \), yields assertion (12) of Lemma 4.

Our reasoning remains valid independently of the order of succession of the points \( O, M, Y', \) and \( X, N, P, Y' \), if account is taken of the signs the lengths of the segments and the quantity \( \Delta \gamma \).

Since \( x > 0 \) and \( \sin \xi \geq 0 \), one may simultaneously pass to the upper and lower limits in (12) for the quantities \( \Delta \gamma/Ax \) and \( \Delta z/Ax \). Along with Lemmas 2 and 3 this leads to the following assertion.

Lemma 5. Suppose that \( OXY \) is a triangle in a metric space, \( \xi \) its upper angle at the vertex \( X \), \( \xi_{\xi_0} \) the lower strong angle between its sides at the same vertex, and \( \gamma \) and \( \xi_0 \) the angles of the plane triangle with the same sides \( x, y, \) and \( z \) corresponding to the vertices \( O \) and \( X \). Then if \( \sin \xi_0 \neq 0 \) we have the inequalities

(14)

\[
\left( \frac{\partial \gamma}{\partial x} \right)_{LL} \geq \frac{\cos \xi - \cos \xi_0}{\sin \xi_0} \cdot \frac{1}{x},
\]

(15)

\[
\left( \frac{\partial \gamma}{\partial x} \right)_{US} \leq \frac{\cos \xi_0 - \cos \xi_0}{\sin \xi_0} \cdot \frac{1}{x}.
\]

In the inequality (15) we are supposing that on \( OX \) there exists at least
one sequence \( X_n \to O \) for which there exist shortest arcs \( X_nY \to XY \), i.e. the concept \( (\partial \gamma / \partial x)_{\text{Lus}} \) exists.

**Lemma 6.** If under the conditions of the preceding lemma \( \xi_0 - \xi \geq \varepsilon > 0 \), then there is a point \( X' \) on \( OX \) lying to the left of \( X \) (\( x' < x \) for it) for which

\[
\gamma(X, Y) - \gamma(X', Y) > M \ln \frac{x}{x'}.
\]

But if \( \xi_{(-\varepsilon)} - \xi_0 \geq \varepsilon > 0 \), and \( (\partial \gamma / \partial x)_{\text{Lus}} \) has a meaning, then there exists a point \( X' \) lying to left of \( X \) for which

\[
\gamma(X', Y) - \gamma(X, Y) > M \ln \frac{x}{x'}.
\]

Here \( M \) is positive and depends only on \( \varepsilon \).

**Proof.** We begin with the first assertion of the lemma. By hypothesis \( \xi_0 - \xi > 0 \), so that \( \xi_0 \neq 0 \). Moreover, \( \xi_0 \neq \pi \), since otherwise in this case there would exist an angle \( \xi \) equal to \( \pi \) and \( \xi_0 = \xi = \pi \). Therefore \( \sin \xi_0 \neq 0 \) and we are justified in using the estimate (14). We shall simplify it by dropping \( \sin \xi \) from the denominator and noting that

\[
\cos \xi - \cos \xi_0 = 2 \sin \frac{\xi_0 + \xi}{2} \sin \frac{\xi_0 - \xi}{2} > 2 \sin^2 \frac{\varepsilon}{2}.
\]

Then we have

\[
\left( \frac{\partial \gamma}{\partial x} \right)_{\text{Lus}} \geq \frac{M}{x},
\]

where \( M = 2 \sin^2 (\varepsilon/2) \). By the definition of the lower derivative we easily obtain from (18) the inequality (16).

The second assertion is proved analogously. But this time we evidently have \( \xi_0 \neq \pi \). Moreover \( \xi_0 \neq 0 \), since if \( \xi_0 = 0 \) we would have \( \xi_0 = \xi = 0 \), which contradicts \( \xi_0 - \xi > 0 \). Further,

\[
\left( \frac{\partial \gamma}{\partial x} \right)_{\text{Lus}} \leq (\cos \xi_{(-\varepsilon)} - \cos \xi_0) \frac{1}{x} \leq - \frac{2 \sin^2 \frac{\varepsilon}{2}}{x},
\]

from which follows inequality (17).

8. **Comparison with a plane triangle.** We denote by \( \delta(T) \) the excess of the triangle \( T \) computed with respect to the upper angles, i.e., the difference

\[
\bar{\alpha} + \bar{\xi} + \bar{\eta} - \pi = \bar{\alpha} + \bar{\xi} + \bar{\eta} - (\alpha_0 + \xi_0 + \eta_0) = \delta(T),
\]

where \( \bar{\alpha}, \bar{\xi}, \bar{\eta} \) are the upper angles of \( T \), and \( \alpha_0, \xi_0, \eta_0 \) are the correspond-
ing angles of the plane triangle with the same sides.

We join the pairs of points $X,Y$ on the sides $AB$ and $AC$ of the triangle $T=ABC$ by shortest arcs $X\overline{Y}$. (If we are dealing with a compact part of a space with an intrinsic metric, then such shortest arcs certainly exist.) For the triangles $AX\overline{Y}$ thus obtained there exist excesses $\delta(AX\overline{Y})$. Consider the quantity\(^{1}\)

\begin{equation}
\bar{\nu}^+_{1}(ABC) = \sup_{X \in AB} \left\{ \inf_{\overline{XY}} \delta(AX\overline{Y}) \right\}.
\end{equation}

We denote the largest of the quantities $\bar{\nu}^+_A, \bar{\nu}^+_B, \bar{\nu}^+_C$ by $\nu^+$. The upper bar reminds us that the excesses are taken with respect to the upper angles and the plus sign recalls the upper limit with respect to the various points $X,Y$. (The quantity $\nu^+$ is not necessarily nonnegative.)

**Theorem 5.** Suppose that $\alpha$ is the upper angle at the vertex $A$ of the triangle $ABC$, and $\alpha_0$ the corresponding angle of a plane triangle with the same sides. We suppose that the triangle $ABC$ lies in a region any two points of which are joined by a shortest arc. Then

\begin{equation}
\alpha - \alpha_0 \leq \bar{\nu}^+_A.
\end{equation}

**Proof.** Suppose the contrary inequality

\begin{equation}
\alpha - \alpha_0 \geq \bar{\nu}^+_A + 3\varepsilon
\end{equation}

were true with $\varepsilon > 0$. Since $ABC$ itself is one of the triangles $AX\overline{Y}$ considered in the expression (20), we may lay out a shortest arc $BC$ such that the following inequality is satisfied:

\begin{equation}
(\alpha - \alpha_0) + (\xi - \xi_0) + (\eta - \eta_0) \leq \bar{\nu}^+_A + \varepsilon.
\end{equation}

Then at least one of the differences $\xi_0 - \xi$, $\eta_0 - \eta$ is not less than $\varepsilon$.

Put $AB = x_0$, $AC = y_0$, $BC = z_0$. If we had $\xi_0 - \xi \geq \varepsilon$, then by Lemma 6 (16) there would be a point $X$ on $AB$ such that $AX = x_1 < x_0$ and

\[\alpha_0(x_0, y_0) - \alpha_0(x_1, y_0) > M \ln \frac{x_0}{x_1}.
\]

If now $\eta_0 - \eta \geq \varepsilon$, then there exists a point $Y$ on $AC$ such that $y_1 < y_0$ and

\[\alpha_0(x_0, y_0) - \alpha_0(x_0, y_1) > M \ln \frac{y_0}{y_1}.
\]

In both cases we have obtained a new triangle with sides $x_1 \leq x_0$, $y_1 \leq y_0$.

\(^{1}\)From the definition used in [13], our definition (20) differs by the presence of an inf with respect to different shortest arcs $X\overline{Y}$ joining the same points $X$ and $Y$. This makes it possible to use the results of Theorem 5 freely.
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(24) \[ \alpha_0(x_0, y_0) - \alpha_0(x_1, y_1) \geq M \ln \frac{x_0 y_0}{x_1 y_1}. \]

Since the angle \( \alpha_0 \) decreases on passing from \( x_0, y_0 \) to \( x_1, y_1 \), inequality (23) holds also for the new triangle and we may repeat the argument. We obtain a triangle with sides \( x_2, y_2 \) for which

\[ \alpha_0(x_1, y_1) - \alpha_0(x_2, y_2) \geq M \ln \frac{x_1 y_1}{x_2 y_2}. \]

Along with (24) this gives us again inequality (24), but with \( x_1, y_1 \) replaced by \( x_2, y_2 \).

Now we consider the greatest lower bound \( t \) of the products \( xy \) for all those \( x, y \) \((0 < x \leq x_0, 0 < y \leq y_0)\) for which

(25) \[ \alpha_0(x_0, y_0) - \alpha_0(x, y) \geq M \ln \frac{x_0 y_0}{xy}. \]

Evidently \( t \geq 0 \). But \( t = 0 \), since for small \( xy \) we would have on the right side of inequality (25) a very large quantity, while on the left there is a difference of quantities not larger than \( \pi \). At the same time \( t > 0 \) is also impossible, since in view of the continuity of \( \ln(x_0 y_0/xy) \) and of \( \alpha_0(x, y) \) the minimum \( t \) would be assumed at some point at which inequality (25) holds. But then one could proceed from these \( x, y \) to a still smaller pair in the same way as in the transition from \( x_1, y_1 \) to \( x_2, y_2 \).

The contradiction thus obtained proves the inadmissibility of the hypothesis (22), i.e., Theorem 5.

Theorem 6. Suppose for each pair of points \( X, Y \) on the side \( AB \) and \( AC \) of the triangle \( ABC \) there is chosen a definite shortest arc \( XY \), while in all cases, convergence from the left (i.e., from the side of the point \( A \)) \( X_n \to X \) or \( Y_n \to Y \) implies \( X_n Y_n \to \overline{XY} \), \( Y_n X_n \to \overline{XY} \). Then the estimate

(26) \[ \nu_{(-)SA} \leq \alpha_{(-)S} - \alpha_0 \]

holds, where \( \alpha_{(-)S} \) is the lower strong angle of the triangle \( ABC \) at the vertex \( A \), and

(27) \[ \nu_{(-)SA} = \inf_{\overline{X Y}} [\delta_{(-)S}(A \overline{XY})]; \]

the excess \( \delta_{(-)S} \) is also measured with respect to the lower strong angles.

Proof. The estimate (26) is established in complete analogy with estimate (21). Suppose that (26) does not hold. Then
\[ \alpha_{(-\gamma)} - \alpha_0 \leq \nu_{(-\gamma)A} - 2\varepsilon \leq \delta_{(-\gamma)A}(ABC) - 2\varepsilon = (\alpha_{(-\gamma)} - \alpha_0) + (\xi_S - \xi_0) + (\eta_{(-\gamma)A} - \eta_0) - 2\varepsilon, \]

while \( \xi_{(-\gamma)A} - \xi_0 \geq \varepsilon \) and \( \eta_{(-\gamma)A} - \eta_0 \geq \varepsilon \). Using Lemma 6 (17), we turn to the triangle \( AX\overline{Y} \) but this time with a larger \( \alpha_0 \). This process may be continued, and as in Theorem 5, where \( \alpha_0 \) decreased, we arrive at a contradiction, arising this time from the increase of \( \alpha_0 \).

9. Two lemmas on plane quadrilaterals.

**Lemma 7.** Suppose that there is a quadrilateral \( ABCD \) situated in the plane, one of whose angles, \( D \), is reentrant (Figure 8). Then under deformation of the quadrilateral with preservation of the lengths of its sides, leading to the rectification of the angle \( D \), we obtain a quadrilateral in which the angles at the vertices \( A \) and \( C \) have increased in comparison with the corresponding angles of the original quadrilateral.

**Proof.** On the extension of the side \( AD \) we lay off \( DC' = DC \). Then \( BC' < BC \), since \( \angle BDC' < \angle BDC \). If now in the triangle \( ABC' \) we increase \( BC' \) to the length of \( BC \), then the angle \( A \) increases. Since the angles \( A \) and \( C \) are on an equal footing, the lemma is proved.

**Lemme 8.** Suppose that in a plane convex quadrilateral with the sides \( c, a, b, b' \) (Figure 9), the angle between the sides \( b' \) and \( b \) is equal to \( \pi - \varepsilon \). Here \( 0 < \varepsilon < \pi \) and \( b' + b \leq c + a \), \( b' \leq c \). Then on rectification of the sides \( b', b \) the angle \( \alpha \) between \( c \) and \( b' \) increases, changing by less than \( 2\varepsilon \):

\[ 0 < \alpha - \alpha' < 2\varepsilon. \]

The notations are clear from Figures 9 and 10. The sides \( b', b \) can be rectified because of the condition \( b' + b \leq c + a \).

We turn to the proof of Lemma 8.
1. With the growth of the angle between \( b \) and \( b' \) the diagonal \( d \)
increases. The angle $\gamma$ increases along with it. Therefore the sum of the remaining two angles of the quadrilateral decreases. Both of them vary the same way as the second diagonal of the quadrilateral. Therefore each of them decreases. In particular

$$0 < \alpha - \alpha'.$$

2. Evidently $\alpha_2 < \varepsilon$. If also $\alpha_1 \leq \varepsilon$, then

$$\alpha - \alpha' = \alpha_1 + \alpha_2 - \alpha' < \alpha_1 + \alpha_2 < 2 \varepsilon.$$

If $\alpha_1 > \varepsilon$ and $\delta_1 \geq \delta_2$, then reflecting the sides $b', b$ in the diagonal $d$, we obtain a nonconvex quadrilateral. Applying Lemma 7 to it we conclude that $\alpha_1 - \alpha_2 < \alpha'$, so that $\alpha_1 - \alpha' < \alpha_2$ and

$$\alpha - \alpha' = \alpha_1 + \alpha_2 - \alpha' < 2 \alpha_2 < 2 \varepsilon.$$

3. It remains to consider the case when $\alpha_1 > \varepsilon$ and $\delta_1 \leq \delta_2$. In fact here we use the condition $b' \leq c$.

Since $\alpha$ and the angle opposite it decrease, each of them varies less than the increase of the two remaining angles. One of the remaining angles increases exactly by $\varepsilon$. Therefore, in order to obtain inequality (28) it is sufficient to show that $\gamma' - \gamma \leq \varepsilon$.

Suppose that $\gamma' - \gamma > \varepsilon$. Then also $\gamma' + \gamma > \varepsilon$. Along with $\gamma' \leq \pi$ and $\gamma < \gamma' - \varepsilon \leq \pi - \varepsilon$ this gives

$$\frac{\varepsilon}{2} < \frac{\gamma' + \gamma}{2} < \pi - \frac{\varepsilon}{2},$$

so that $\sin(\gamma' + \gamma/2) > \sin(\varepsilon/2)$. We also note that from the conditions $b' \leq c$, $\alpha_1 > \varepsilon > \alpha_2$, $\delta_1 \leq \delta_2$ it follows that $b < a$.

Now we turn directly to the comparison of the angles $\gamma$ and $\gamma'$. We have:

$$d_2 = a_2 + c_2 - 2ac \cos \gamma,$$

$$d'_2 = a_2 + c_2 - 2ac \cos \gamma',$$

so that

$$\cos \gamma - \cos \gamma' = \frac{d'^2 - d^2}{2ac}.$$

From the triangle $b'bd$ and the equality $d' = b' + b$ we find that

$$d'^2 - d^2 = 4b'b \sin^2 \frac{\varepsilon}{2}.$$
Then the preceding equation takes the form
\[ \sin \frac{\gamma' + \gamma}{2} \sin \frac{\gamma' - \gamma}{2} = \frac{b' b}{c a} \sin^2 \frac{\varepsilon}{2}. \]

But \( \sin((\gamma' + \gamma)/2) > \sin(\varepsilon/2) \), \( b' < c, \ b < a \). This leads us to the inequality
\[ \sin \frac{\gamma' - \gamma}{2} < \sin \frac{\varepsilon}{2}, \]
where \( \varepsilon \) and \( \gamma' - \gamma \) lie between 0 and \( \pi \). This inequality contradicts the hypothesis \( \gamma' - \gamma > \varepsilon \).

Thus Lemma 8 is completely proved.

Aside from these two lemmas, we need the following elementary criterion for convergence.

**Lemma 9.** If in the sequence \( x_1, x_2, \ldots \) all the \( x_i > M \) and for any \( \varepsilon > 0 \) there exist \( N, N_i \) such that if \( m > n > N \) and \( m > N_i \)
\[ x_m < x_n + \varepsilon, \]
then the sequence \( x_1, x_2, \ldots \) converges.

**Proof.** Since \( x_i > M \), \( \lim \inf x_i \) exists. Suppose that \( n > N \) and \( x_n < \lim \inf x_i + \varepsilon \). For \( m > n \) and \( m > N_i \) we have:
\[ x_m < x_n + \varepsilon \leq \lim \inf x_i + 2\varepsilon. \]

Thus \( x_m \) are bounded above and
\[ \lim \sup x_i \leq \lim \inf x_i + 2\varepsilon. \]

Because of the arbitrarily small size of \( \varepsilon \) it therefore follows that \( \lim \sup x_i \leq \lim \inf x_i \) which gives the convergence of the sequence \( x_i \).

10. **Sufficient conditions for the existence of the angle.**

**Theorem 7.** Suppose that the shortest arcs \( L \) and \( M \) issue from the point \( O \). If arbitrary points \( X \in L \) and \( Y \in M \) close to \( O \) can be joined by a shortest arc \( XY \) such that for the triangles thus cut off the quantities \( \bar{v}_0(OXY) \) remain less than any positive number as \( X \) and \( Y \) approach \( O \), then there exists between \( L \) and \( M \) at the point \( O \) a definite angle.

**Proof.** By the fundamental Theorem 5 we have:
\[ \bar{\alpha} - \gamma \leq \bar{v}_0, \]
or in other words
\[ \bar{\alpha} \leq \gamma + \bar{v}_0. \]

Passing on the right to the lower limit of the value of \( \gamma \) for \( X, Y \to O \),
we have:

$$\bar{\alpha} \leq \lim \inf (X,Y) = \alpha$$

Thus $\alpha_- = \bar{\alpha}$ and a definite angle exists.

Remarks. 1) The conditions of Theorem 7 are not necessary for the existence of an angle. For example, if in an acute-angled plane sector with sides $L$ and $M$ we make a series of circular holes into which we insert cylindrical protuberances concentrated at the vertex, as depicted on Figure 11, then it is easy to obtain a surface on which between the shortest arcs $L$ and $M$ there exists an angle, but for a series of pairs $X,Y \to O$ the quantities $\delta(OXY)$, and along with them also $\bar{\nu}(OXY)$ will remain significantly positive.

2) If each point of the space has a neighborhood $G$ in which for any triangle $T$ the sum of the upper angles is not larger than in the triangle with the same sides on a surface of positive curvature $T$, then such a space is called a space with curvature not larger than $K$. A simple consequence of Theorem 7 is the following assertion:

In a space with curvature not larger than $K$, there exists an angle between any two shortest arcs.\(^3\)

Indeed, in this case $\bar{\delta}_K \leq 0$. But as $X,Y \to O$ the difference between $\bar{\delta}_K$ and $\delta$ tends to zero. Therefore $\bar{\nu}^+(OXY)$ remains less than any $\varepsilon < 0$, so that the conditions of Theorem 7 are satisfied.

Theorem 8. Suppose that $L$ and $M$ are shortest arcs issuing from the point $O$. If each point $X \in L$ can be joined to each point $Y \in M$ by a shortest arc $XY$, such that for any triple of points $X,Y,Z$ distinct from $O$ (of which $Y$ lies on one, and $X$ and $Z$ lie on the other of the shortest arcs $L$, $M$) the quantities $\bar{\nu}^+(XYZ)$ for the resulting triangles remain less than any positive number when $X,Y,Z \to O$, then there exists a definite angle between $L$ and $M$ at the point $O$.

We recall that $\bar{\nu}^+ = \max (\bar{\nu}^+, \bar{\nu}^+, \bar{\nu}^+)$.

\(^3\) We assume of course, that we are dealing with a space with an intrinsic and that close to each point we may join any pair of points by shortest arcs.
Proof. We begin with a simple remark. If we consider an arbitrarily small neighborhood $G(\varepsilon)$ of the point $O$, so small that in it $\bar{\nu}^+(XYZ) < \varepsilon$, then for the angles $\xi$ and $\eta$ at the vertices $X, Y$ in the triangle $\bar{XYZ}$ the inequalities

$$\xi_0 > \xi - \varepsilon, \quad \eta_0 > \eta - \varepsilon$$

are valid, where $\xi_0$ and $\eta_0$ are the corresponding angles in the plane triangle with the same lengths of its sides. The same estimates with account taken of the sign remain valid also for the triangle $\bar{OXY}$. Indeed, for $\bar{XYZ}$ the assertion follows immediately from Theorem 5:

$$\xi - \xi_0 \leq \bar{\nu}_x^+(XYZ) \leq \bar{\nu}^+(XYZ).$$

For the triangle $\bar{OXY}$ itself the inequality $\xi - \xi_0 \leq \varepsilon$ is obtained from the first of the inequalities (29) as $Z \to O$, since $\xi_0$ continuously depends on $Z$. For $\eta$ everything is analogous.

Now we turn to the question of the existence of a limit for the angles $\gamma$.

Suppose that $X \in L$, $Y \in M$ and $X, Y \in G(\varepsilon)$. Choose on $L$ and $M$ arbitrary points $X', Y'$ closer to $O$ than the closest of the points $X, Y$. We assert that for such $X', Y'$

$$\gamma' - \gamma < 8\varepsilon,$$

where $\gamma' = \gamma(X', Y')$ and $\gamma = \gamma(X, Y)$.

In order to establish this inequality, we join the points $X, Y; X', Y'$ and $Y', X$ by shortest arcs, chosen according to the conditions of the theorem (Figure 12), and consider plane triangles with the sides $OY'$,

![Figure 12.](image)

$OX', X'Y'$ and $X'Y', Y'X, X'X$, which we add to one another along the side $X'Y'$ (Figure 13).

If in the plane quadrilateral $OY' XX'$ the angle at $X'$ is reentrant,
then from Lemma 7 the angle $\gamma'$ increases on the rectification of $OX'X$, so that for the new value $\gamma_*$ we have

$$\gamma' - \gamma_* < 0.$$  

If $OX'X$ is a straight line, we have here the equality sign.

If the angle at $X'$ is external, still it cannot be much less than $\pi$. Indeed, with the notation shown in the drawing, we have from (29) and (30) that:

$$\xi_{01} > \xi_1 - \varepsilon, \quad \xi_{02} > \xi_2 - \varepsilon,$$

from which

$$\xi_{01} + \xi_{02} > \xi_1 + \xi_2 - 2\varepsilon \geq \pi - 2\varepsilon.$$  

For the quadrilateral $OY'XX'$ the conditions of Lemma 8 are satisfied, from which we now obtain

$$\gamma' - \gamma_* < 4\varepsilon.$$  

This last inequality is a fortiori true also in the preceding cases.

Repeating exactly the same argument the quadrilateral $OXYY'$, we prove that

$$\gamma' - \gamma < 8\varepsilon.$$  

From the inequality just obtained it already follows that the limit of the angles $\gamma$ exists as $X,Y \rightarrow O$. It is sufficient to make use of the criterion for convergence proved in Lemma 9. Theorem 8 is proved.  

Remark. After the existence of the angle has been proved, it is not difficult to verify that under the conditions of Theorem 8 the conditions of Theorem 7 are also satisfied. We will not present the arguments here, but merely note that they entirely coincide with the method of proof of the relation (31) in Lemma 11 below.

5. Angle of a sector. In this section we consider a space which is a two-dimensional manifold of bounded curvature in the sense of the definition of subsection 6 of Chapter I.

11. Sector. Suppose that simple curves $L$ and $M$ issue from the point $O$ and have no common points other than $O$. Each sufficiently small neighborhood $U$ of the point $O$ that is homeomorphic to the closed disc is divided by the curves $L$ and $M$ (more precisely by their portions from $O$ to the first intersection with the boundary of $U$) into two bounded sectors $C'(U), C''(U)$. Their properties, connected with the arbitrarily small neighborhood of the point $O$, do not depend on the specific choice
of $U$ and thus relate simply to the sectors $C', C''$.

Let us make this concept of sector more precise. We shall say that $C'(U_1)$ and $C'(U_2)$ are equivalent if there exist $C'(U_3) \subset C'(U_1), C'(U_4) \subset C'(U_2)$. The sector $C'$ is a class of equivalent bounded sectors $C'(U)$. In this sense $L$ and $M$ form two sectors, with $L$ and $M$ their sides and $O$ the vertex. It is convenient to regard two simple curves coinciding on a portion close to $O$ as forming two sectors; one of them consists only of its sides.

The phrase "a point lies in a sector" has a meaning only for a fixed neighborhood $U$. But references to $U$ can be dropped if we are considering a point arbitrarily close to $O$. In particular, the phrase: "$X_n \to O$ in the sector $C$" has an obvious meaning as "the curve $N$ leaves $O$ in the sector $C$".

A definite choice of one of the two sectors may be made by fixing in advance the neighborhood $U$, or by indicating a curve which issues from its vertex and is distinct from the sides of the sector, or by indicating a sequence of points which converge in the sector to its vertex.

In what follows we shall be dealing almost exclusively with sectors whose sides are shortest arcs.

12. The leftmost shortest arc. Suppose that $C(U)$ is a bounded sector homeomorphic to the closed disc. Let $O$ be its vertex, and let the shortest arcs $L$ and $M$ be its sides. Suppose that the points $X \in L$, $Y \in M$ distinct from $O$ can be joined by at least one shortest arc $XY$ in $C(U)$. This shortest arc divides $C(U)$ into two parts (the curve $XY$ itself belongs to both of them). One of these parts is the triangle $OXY$.

**Lemma 10.** Suppose that $X$ and $Y$ are joined in $C(U)$ by two noncoinciding shortest arcs $N_1$ and $N_2$, neither of which lies to the left of the other. Then they may be joined by a third shortest arc $N_3$ lying to the left of $N_1$, $N_2$, i.e., different from them and lying in each of the two triangles $OXY$ which they cut off.

**Proof.** We shall measure off on the shortest arc $N$ the length of the arc $s$ to the point $X$. Each common point of $N_1$ and $N_2$ corresponds to the same value of the parameter $s \in [0,l]$, for otherwise the shortest path of length $l$ from $X$ to $Y$ could be further contracted. To the set of common points of $N_1$ and $N_2$ there corresponds a closed set on $[0,l]$. Since $N_1 \neq N_2$, it does not exhaust $[0,l]$. The complement consists of a countable collection of open intervals. Starting with $N_1$, we shall suc-
cessively on each of these intervals replace a section of $N_1$ by the leftward of the two available paths. As a limit we obtain a shortest arc $N_3$ going to the left of $N_1$ and $N_2$.

We shall say that $XY$ is the leftmost shortest arc if there is no other shortest arc $XY$ in the triangle $OXY$. If one of the points $X, Y$ coincides with $O$, then the leftmost shortest arc will be taken to be a section of the side of the sector.

**Theorem 9.** If in $C(U)$ the points $X$ and $Y$ are joined by at least one shortest arc, then there exists a leftmost shortest arc $XY$. If the points $X_n$ converge to $X$ to the left along $L$ (i.e., $X_n \rightarrow X$ and $X_n \in OX \subset L$) and there exist shortest arcs $X_nY$ lying in $C(U)$, then the leftmost shortest arcs $X_nY$ converge to the leftmost shortest arc $XY$.

**Proof.** Consider all shortest arcs $N$, going from $X$ into $Y$ in the sector $C(U)$. Each of them leaves in $C(U)$ an open set to the right of itself, $G(N)$. Consider the set $G = \bigcup_N G(N)$. Because of the countability of the basis of a two-dimensional manifold there exists a sequence of open basis sets $g_1, g_2, \ldots$, each of which lies in at least one $G(N)$ and which together exhaust $G$. For each number $i$ there is a shortest arc $N_i$, leaving in $C(U)$ a set $g_i$ to the right of itself.

Suppose that $N_1 = N_1', N_2$ is a shortest arc going to the left of $N_1'$ and $N_2'$, and that $N_3$ goes to the left of $N_2$ and $N_2'$, and so forth. By Lemma 10, there exist such shortest arcs.

From among the shortest arcs $N_1, N_2, \ldots$, one may select a convergent subsequence. It is easily verified that its limit is the leftmost shortest arc $XY$. It is unique in view of Lemma 10.

Now suppose that the $X_n$ converge to $X$ leftward along $L$ and that $X_nY$ is the most leftward shortest arc. Selecting a subsequence, we may suppose that $X_nY$ converges to some shortest arc $XY$. If now $XY$ did not coincide with the leftmost shortest arc $XY$, it would fall to the right of $XY$. Then for large $n$ the shortest arc $X_nY$ would also fall to the right of $XY$, intersecting $XY$, and would not be the leftmost shortest arc.

Theorem 9 is proved.

We note that the property of $XY$ of being the most leftward shortest arc does not change when $C(U)$ is extended by varying the region $U$. Therefore the following makes sense: “the sequence of leftmost shortest arcs $X_nY_n$ in the sector $C$".
13. **The existence of an angle between the sides of a sector convex with respect to its boundary.** Many of the proofs in this chapter, as we shall see later, are true also under wider conditions. This statement holds in particular for Theorem 10, which we shall now prove only for sectors of a special sort. Suppose that the sector $C$ is formed of the shortest arcs $L$ and $M$, either coinciding or not having any common points close to the vertex $O$ other than $O$. We shall say that the sector $C$ is **convex relative to its boundary** if no points $X \in L$ and $Y \in M$ close to $O$ can be joined by an arc $XY$ passing outside $C(U)$ which is shorter than $XO + OY$.

The following assertions are valid in an obvious way. The property of a sector $C$ of being convex relative to its boundary does not depend on the choice of $C(U)$. In a sector convex relative to its boundary two points $A, B \in C(U)$ sufficiently close to $O$ may be joined by a shortest arc going in $C(U)$. In this sense each sector convex relative to its boundary is convex. If $L$ and $M$ extend one another, forming one shortest arc, then the sectors formed by them are convex relative to their boundaries. If the sector $C$ is divided by a shortest arc passing inside it into two sectors, then both of these sectors are convex relative to the boundary, given that $C$ was.

**Theorem 10.** **Between the sides of a sector $C$ convex relative to its boundary there always exists a definite angle.**

**Proof.** If the sides of the sectors, the shortest arcs $L$ and $M$, coincide, then they form a null angle. If on any small section around $O$ they extend one another, forming one shortest arc, then they form the angle $\pi$.

We consider the remaining case and show that here the sufficient conditions for the existence of an angle, as formulated in Theorem 8, will be satisfied. As the system of shortest arcs $XY$ we choose the leftmost shortest arcs in $C(U)$. Suppose that $X, Z \in L$, $Y \in M$ are points different from one another and from $O$, sufficiently close to $O$. We shall show that for the triangles $XYZ$, for any $\varepsilon > 0$, the requirement $\nu^+ < 2\varepsilon$ is satisfied, for $X, Y, Z$ sufficiently close to $O$.

In view of the fact that $XY$ and $YZ$ are leftmost shortest arcs, the triangle $T = XYZ$ can have only one of the six topological structures depicted in Figure 14. In view of the fact that the shortest arcs $L$ and $M$ are not extensions of one another in any section, $T$ does not contain the point $O$. If for such a triangle $\nu^+ \geq 2\varepsilon$, then in $T$ by joining the
points on the two sides by the leftmost shortest arc in it, we can cut off a triangle $T''$ for which $\bar{\delta} > \varepsilon$. Once $\bar{\delta}(T'') > 0$, the triangle $T''$ can have only one of the first four structures depicted on Figure 14. The excess $\bar{\delta}$ does not decrease if we reject from $T'$ the "outside tails". Thus we obtain a triangle $T'''$ lying in $T$ and homeomorphic to the disc, for which $\bar{\delta}(T''') > \varepsilon$. It is easily verified that $T'''$ is also a triangle convex relative to its boundary.

If now arbitrarily close to $O$ there existed triangles $XYZ$ for which $\nu^+ \geq 2\varepsilon$, we would have close to $O$ an arbitrarily large number of non-overlapping simple triangles $T''''$, for each of which $\bar{\delta} > \varepsilon$, which would contradict the condition of bounded curvature.

Theorem 10 is thus proved.

In the triangle $OXY$ all three sectors are convex relative to the boundary. Therefore the triangles $OXY$ have definite angles, and one may speak simply of the excesses $\delta(OXY)$. Later on (in the proof of Lemma 11 of Chapter III) we shall need the following assertion.

**Lemma 11.** Under the conditions of Theorem 10

$$\lim_{x,y \to 0} \delta(OXY) = 0.$$ 

**Proof.** We shall show first that for any $\varepsilon > 0$ and for $X,Y$ sufficiently close to $O$

$$\delta(OXY) < 5\varepsilon. \quad (31)$$

If $XY$ passes through $O$, then close to $O$ all the $\delta(OXY) = 0$. Consider the case when all the leftmost shortest arcs do not pass through $O$.

Suppose that the points $X,Y$ are close to $O$. Then from Theorem 10 we may suppose that $|\alpha - \alpha_0| < \varepsilon$, where $\alpha$ is the angle of the triangle $OXY$ at the vertex $O$, and $\alpha_0$ is the angle of the plane triangle with the same sides. On the sides $OY$ and $OX$ we mark off the points $Z', Z''$ (Figure 15),
close enough to $O$ that $|\xi'_0 - \xi_0| < \varepsilon$ and $|\eta''_0 - \eta_0| < \varepsilon$, where $\xi_0, \eta_0$ are the angles of the plane triangle with the sides $OX, OY, XY$, and $\xi'_0, \eta''_0$ are the corresponding angles in the plane triangles with the sides $Z'X, Z'Y, XY$ and $Z''X, Z''Y, XY$.

Inside the sector in question we lay off the leftmost shortest arc $Z'X$. For the triangle $Z'XY$ thus obtained, as in the proof of Theorem 10, we have $\phi^+(Z'XY) < \varepsilon$ for $X, Y$ sufficiently close to $O$. Analogously $\phi^+(Z''XY) < \varepsilon$.

Adding the inequalities

$$\alpha - \alpha_0 < \varepsilon,$$
$$\xi - \xi'_0 \leq \phi^+(Z'XY) < \varepsilon,$$
$$\xi'_0 - \xi_0 < \varepsilon,$$
$$\eta - \eta''_0 \leq \phi^+(Z''XY) < \varepsilon,$$
$$\eta''_0 - \eta_0 < \varepsilon,$$

we obtain

$$\alpha + \xi + \eta - (\alpha_0 + \xi'_0 + \eta_0) = \delta(OXY) < 5\varepsilon.$$

Now we shall show that for $X, Y$ sufficiently close to $O$,

$$\delta(OXY) \geq -5\varepsilon. \tag{32}$$

Suppose that there exist pairs of points $X, Y$ arbitrarily close to $O$ and for which $\delta(OXY) < -5\varepsilon$. We assert that then for any $X, Y$ sufficiently close to $O$ we have

$$\delta_-(OXY) < -3\varepsilon. \tag{33}$$

Indeed, suppose that $OXY$ is a small triangle, and that $OAB$ is even smaller and that $\delta(OAB) < -5\varepsilon$. Introduce the notation for angles depicted on Figure 16. Then for $X, Y$ sufficiently close to $O$, we have

$$\angle 1 + \angle 2 + \angle 9 - \pi < -5\varepsilon, \tag{by the choice of \varepsilon}$$
$$\angle 3 + \angle 4 + \angle 8 - \pi < \varepsilon, \tag{by Theorem 8}$$
$$\angle 5 + \angle 6 + \angle 7 - \pi < \varepsilon,$$

$$\pi \leq \angle 2 + \angle 3,$$
$$\pi \leq \angle 7 + \angle 8 + \angle 9, \tag{as in Theorem 10}$$
$$\angle 10 \leq \angle 4 + \angle 5.$$
Adding these inequalities, we obtain inequality (33).
By Theorem 10 for $X, Y$ close to $O$, we have $|\alpha - \alpha_0| < \varepsilon$, so that from $\alpha - \alpha_0 + \xi - \xi_0 + \eta - \eta_0 < -3\varepsilon$ it follows that
$$\xi - \xi_0 + \eta - \eta_0 < -2\varepsilon,$$
i.e., $\xi_0 - \xi > \varepsilon$ and $\eta_0 - \eta > \varepsilon$.

But from Lemma 6 the validity of this condition generates a process of decreasing the angle $\alpha_0$ under shifts of $X$ or $Y$ toward the point $O$, while the fact that (33) remains true leads to an unbounded decrease of $\alpha_0$ similar to that which we had in the proof of the fundamental Theorem 5 on the angles of a triangle.

This contradiction proves Lemma 11.

14. The angle of a sector. Later on, in Chapter IV, we shall verify that in a two-dimensional manifold of bounded curvature there exists an angle between any two shortest arcs, issuing from one point. We shall suppose that the sector $C$ with vertex $O$ generated by the shortest arcs $L$ and $M$ divides the shortest arcs issuing from $O$ into several successive sectors. We form the successive sum of the angles between the sides of these sectors. The least upper bound of these sums for all possible decompositions of the sector $C$ is called the angle of the sector $C$.

It is not assumed in advance that this angle is finite. The usual angle of a sector will be denoted by $\tilde{\alpha}$, and the angle between its sides by $\alpha$.

Even one shortest arc issuing from the point $A$ divides its neighborhood into two sectors. Consider all possible decompositions of the neighborhood of the point $A$ into successive adjacent sectors. Taken with all such decompositions the least upper bound of the sum of the angles between the sides of these sectors is called complete angle around the point $A$. We shall denote it by $\theta(A)$ or simply $\theta$. The finiteness of $\theta$ is not assumed in advance.

In Chapter IV we shall verify the following assertions. The complete angles $\theta(A)$ and the angle of the sector $\tilde{\alpha}$ are always finite. The angle of the sector is additive. If the shortest arcs $L$ and $M$ form two sectors, then $\theta = \tilde{\alpha}_1 + \tilde{\alpha}_2$, $\alpha = \min[\pi, \tilde{\alpha}_1, \tilde{\alpha}_2]$. If $\theta > 0$, then the neighborhood of a point admits a decomposition into convex sectors. If $\theta > \pi$, then a neighborhood of the point admits a decomposition into sectors convex relative to the boundary. If the decomposition into sectors of the indicated types is possible, then the values of $\tilde{\alpha}$ and $\theta$, defined by these decompositions alone, coincide with the usual values of $\tilde{\alpha}$ and $\theta$. 
At present we know only about the existence of an angle between the sides of a sector convex relative to the boundary. Therefore we shall speak of the angle of a sector only for sectors which can be decomposed into sectors convex relative to the boundary, and define the value of $\alpha$ only for such a decomposition. Analogously, the complete angle around a point is defined up to now only for points whose neighborhoods may be decomposed into sectors convex relative to the boundary, and the value of $\theta$ is defined only with respect to such decompositions.

**Lemma 12.** For a sector $C$ admitting a decomposition into sectors convex relative to the boundary,

$$\alpha = \min [\pi, \tilde{\alpha}],$$

where $\tilde{\alpha}$ is the angle of the sector and $\alpha$ the angle between its sides.

**Proof.** If the sides $L$ and $M$ of the sector $C$ are extensions of each other, then $\alpha = \pi$, $\tilde{\alpha} \geq \pi$ and (34) is true.

Suppose that $L$ and $M$ are not extensions of each other. Then for any decomposition $C$ by shortest arcs forming sectors convex relative to the boundary the conditions of Theorem 3 are satisfied. Therefore for any such decomposition the sum of the successive angles between the shortest arcs is equal to $\alpha$ and $\tilde{\alpha} = \alpha \leq \pi$. This proves Lemma 12.

15. Points through which shortest arcs pass. Of basic importance are the points which have at least one shortest arc passing through them (it is assumed that the point is an interior point for this shortest arc). On the surface of a circular cone, not even one shortest arc passes through the vertex (conical point). We can also construct a smooth convex surface through some points of which there pass no shortest arcs.

In small neighborhoods any two points can be joined by shortest arcs. Therefore points which have shortest arcs passing through them constitute the bulk of the points. They are everywhere dense. Moreover, on each simple curve the points through which a shortest arc passes are everywhere dense on the curve. (Indeed, any point $O$ on a curve can be enclosed by a small neighborhood homeomorphic to the disc. The arc of the curve divides this neighborhood. Taking points close to $O$ on both sides of the curve and joining them by shortest arcs, we discover on the curve a point close to $O$ through which there passes a shortest arc.)

If we start the construction with points through which shortest arcs pass, join these points by shortest arcs and introduce new points on their intersections, we never leave the class of points through which
there pass shortest arcs.

Operating with such points is convenient in view of the following theorem.

**Theorem 11.** *If through the point O there passes at least one shortest arc, then for any decomposition of a neighborhood of the point O into sectors convex relative to the boundary the sum of the angles between the sides of these sectors is not less than $2\pi$.*

**Proof.** The indicated decompositions exist: one of them gives the branches $M$ and $N$ of a shortest arc passing through $O$. Suppose that we have an arbitrary decomposition of a neighborhood of the point $O$ into sectors convex relative to the boundary by shortest arcs $L_1, \ldots, L_n$.

If the branch $M$ intersects one of the shortest arcs $L_i$ at a point distinct from $O$, then the initial portion of $M$ may be replaced by the segment $L_i$. Otherwise, $M$ passes between some shortest arcs $L_i, L_{i+1}$. Then from the method of proof of Lemma 12 it follows either these two shortest arcs are extensions of each other or else that $M$ can be included in the set of dividing curves without changing the sum of the angles between the sides of the sectors. Analogous remarks are valid for the branch $N$.

Thus we either find among the shortest arcs $L_i$ two which are extensions of each other or, without changing the sum of the angles, we arrive at a division in which there is such a pair of shortest arcs. But then on each side of the shortest arc passing through $O$ the sum of the angles is not less than $\pi$. Theorem 11 is proved.
CHAPTER III

Approximation by Polyhedral Metrics

In this chapter it is proved that the metric of every manifold of bounded curvature admits locally uniform approximation by polyhedral metrics, the integral absolute curvatures of which are collectively bounded. Precise formulations are given in Theorem 10 of this chapter and in Remark 2 at the end of the chapter.

The entire argument applies to two-dimensional manifolds of bounded curvature.

1. Neighborhood of a point. In §1 it will be established that every point has an arbitrarily small neighborhood which is an absolutely convex polyhedron homeomorphic to the disc, with a small perimeter and with vertices lying at points through which there pass shortest arcs.

1. Types of convex figures. A convex figure is a point set to which, along with each pair of its points, there belongs at least one shortest arc joining these points. We shall call a figure fully convex if each two of its points can be joined by shortest arcs and all such shortest arcs belong to the figure. The following assertion is obvious.

Lemma 1. The intersection of a convex and a fully convex figure is convex. The intersection of two fully convex figures is fully convex.

The intersection of two simply convex figures may fail to be convex. For example, in the space represented by the surface of a sphere two meridians by themselves are convex figures, but their intersection a two-point set consisting of the two poles, is not even connected and certainly not convex.

Suppose that the figure $\Phi$ lies in a region $G$ homeomorphic to the open disc, and that it is bounded by a simple closed rectifiable curve whose perimeter is less than four times the distance from $\Phi$ to the boundary of $G$. We shall say that the figure $\Phi$ is convex relative to its boundary if no two points $X,Y$ of the boundary of $\Phi$ can be joined by an arc $XY$ lying outside $\Phi$ which is shorter than the piece $XY$ of the boundary of the figure $\Phi$ enclosed by that arc.
The following assertion is obvious.

**Lemma 2.** *If a figure convex relative to its boundary is divided into two pieces by a shortest arc which joins two points of the boundary of that figure and which lies inside that figure with the exception of its endpoints, then each of the pieces thus obtained becomes in its turn convex relative to the boundary.* (The dividing curve itself is counted here in each of the pieces of the figure.)

A figure which is fully convex and convex relative to its boundary will be called *absolutely convex.*

**Remark.** On the plane every bounded convex figure is at the same time fully convex and convex relative to its boundary. But even on the surface of a right circular cone, for figures bounded by segments of two generators, this may not be so.

Evidently a figure convex relative to its boundary is convex.

The property of a figure of being convex relative to its boundary or of being absolutely convex is connected with the choice of the region $G$. These properties are evidently maintained on enlarging the region $G$.

Lemmas 1 and 2 show that the property of a figure of being fully convex is "inherited" under intersection of figures, and the property of being convex relative to the boundary is inherited under cutting of the figure by a shortest arc.

The property of being absolutely convex may in the second case be destroyed.

The concept of convexity relative to the boundary may be extended to a figure bounded by a closed curve which has multiple points but admits an infinitely small deformation into a simple closed curve. In this case Lemma 2 is true for any section of a figure convex relative to its boundary by a shortest arc in it.

2. **Types of triangles.** We shall call a figure consisting of three distinct points (the vertices) and three shortest arcs joining these points pairwise (the sides) a *triangle.* Suppose that a triangle lies in a region $G$ homeomorphic to the disc. We shall call it an *inflatable* triangle if its contour, as a closed curved line, admits an arbitrarily small deformation into a simple closed curve. A fixed point $X$ will be called an *interior* point of an inflatable triangle if for all sufficiently small deformations

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1 By smallness of the deformation we have in mind closeness of the curves in the sense of the distance between curves.
of the contour of the triangle into a simple closed curve the point $X$ lies inside this curve. For an inflatable triangle its interior points will be counted with the triangle.

A finite system of inflatable triangles is called a system of nonoverlapping triangles if their contours admit arbitrarily small simultaneous deformations into simple closed curves such that the deformed "triangles" have no common interior points.

A triangle $T$ will be said to be reduced if it lies in a region $G$ homeomorphic to the disc and no two of its sides have common points other than a common vertex or a common initial section adjacent to that vertex.

Obviously one may pass from any triangle in $G$ to a reduced one. If there are several pairs of sides with common points it suffices to replace a terminal section of one of the sides by a segment of the other side of the triangle.

Obviously every reduced triangle is an inflatable triangle, so that the concept of its interior points makes sense. Interior points are added to the triangle. The concept of a system of nonoverlapping reduced triangles also makes sense.

Every reduced triangle may be classified as one of 12 types according to its structure, as illustrated in Figure 17. The first of these is a triangle homeomorphic to a disc. The second thru fourth are triangles with "exterior tails". The fifth consists only of exterior tails. The sixth is a triangle degenerated into a segment. The seventh thru twelfth are triangles with "interior tails". Examples of triangles of each of these types may be constructed on nonconvex manifolds.

Considering a reduced triangle as a point set, one may speak of its convexity or nonconvexity. Taking into account the fact that a contour has been chosen (of its three sides), we may speak of its convexity or nonconvexity relative to the boundary. (This remark holds for all inflatable triangles.) Evidently a reduced triangle which is convex relative to its boundary can have no interior tails. As in subsection 2 of Chapter I, a triangle which is convex relative to its boundary and homeomorphic to a disc is called a simple triangle.

Let us make another remark about the above process of reduction of

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2 Note the specific character of the concept of "interior point of a triangle". For example, for a triangle with an "interior tail", a vertex of the triangle lying inside that triangle is not an interior point.
triangles. A system of shortest arcs will be called a *system without superfluous intersections* if each pair of shortest arcs of this system either do not intersect or have one common point, or if their common part reduces to a single segment (shortest arc) common to both.

![Figure 17](image)

If two points in general admit joining by a shortest arc, then the latter may always be chosen so that no superfluous intersections will be created with an already existing finite system of shortest arcs which itself did not already have superfluous intersections. We shall make constant use of this obvious fact.

3. **Neighborhood with finite perimeter.**

**Lemma 3.** Every point \( O \) has an arbitrarily small neighborhood homeomorphic to the disc and bounded by a simple closed polygon with vertices at points through which there pass shortest arcs.

The perimeter of this region thus has finite length. It will suffice to describe the plan of the proof. To the point \( O \) there corresponds an arbitrarily small neighborhood \( G \) homeomorphic to the disc. In \( G \) we may encircle \( O \) with a simple closed curve \( \Gamma \). If on \( \Gamma \) we take points sufficiently close to one another through which there pass shortest arcs that do not create superfluous intersections, we obtain a polygon enclosing \( O \). Rejecting superfluous loops from this polygon, we obtain the simple polygon required by Lemma 3, enclosing \( O \) and *a fortiori* having vertices at points through which there pass shortest arcs.

4. **Concerning the shortest loop.** Consider a neighborhood \( G \) of the point \( O \) homeomorphic to the disc, and in it a closed neighborhood \( V \) of the point \( O \) bounded by a simple closed curve \( \Gamma \). Suppose that \( \gamma \) is a simple closed polygon encircling \( O \) and lying inside the region bounded by the curve \( \Gamma \).

**Lemma 4.** If the length of \( \gamma \) is sufficiently small then among the simple closed curves encircling \( \gamma \) and lying in \( V \) there exists at least one which is shortest, and it lies inside \( V \). Each of these shortest curves bounds a region convex relative to its boundary. Among these shortest curves is the “most interior” and the “most exterior”; the most exterior of these curves
bounds an absolutely convex region. Each of the shortest curves is a closed geodesic or a closed geodesic polygonal line whose vertices, if they are not interior points of a geodesic, lie at the vertices of the polygonal line $\gamma$.

We shall give only a rather detailed outline of the proof of this lemma.

1. For each point $X \in \Gamma$ we consider the greatest lower bound $\varepsilon(X)$ of the lengths of simple closed curves passing through $X$, enveloping $O$ and lying in $\bar{G}$. This quantity is positive since $O \in \Gamma$. Moreover, $\varepsilon(X)$ depends continuously on the point $X$ on $\Gamma$. We denote by $\varepsilon$ the smaller of the following two quantities: twice the distance from $F$ to the boundary of the region $G$ and the minimum of $\varepsilon(X)$ with respect to all $X \in \Gamma$. Evidently $\varepsilon > 0$. We shall suppose also that the length of $\gamma$ is less than $\varepsilon$.

2. The class of simple closed curves lying in $V$ and enveloping $\gamma$ may be supplemented by curves which appear as limits of sequences of curves in that class with bounded length. In this extension of the class, there exists, in view of the compactness of $V$, at least one curve $L$ of smallest length. From its minimal property we see that it is itself a simple closed curve. (For otherwise it could be shortened.) This curve also envelops $\gamma$. Evidently, the length of the shortest curve is not greater than the length of $\gamma$ (we consider that $\gamma$ encloses itself) and therefore is less than $\varepsilon$. Taking account of the choice of $\varepsilon$, we conclude that that curve does not encounter $\Gamma$ and therefore lies entirely inside $V$.

3. Each minimal curve $L$ encircles a region convex relative to its boundary. Suppose the contrary. Suppose that two points of this curve can be joined by an arc which passes outside the region bounded by $L$ and which is moreover shorter than the corresponding segment of $L$. Evidently this arc will not go out of $G$, since otherwise it would be longer than twice the distance from $\Gamma$ to the boundary of $G$ and thus longer than $\varepsilon$, i.e. longer even than the entire curve $\gamma$. Since it passes inside $G$, this arc along with the part of the curve $L$ forms a simple loop enveloping $\gamma$ and shorter than $L$. But the loop cannot intersect $\Gamma$ since its length is less than $\varepsilon$, which means that it lies inside $V$. But this leads to a contradiction with the minimal property of $L$.

4. Of two curves, of which the first lies in a closed region bounded by the second curve, it is natural to consider the first as interior and the second as exterior. The existence of the “most interior” and the “most exterior” of the minimal curves in question is established in exactly the same way as was done for the “leftmost” shortest arcs in subsection 12 of Chapter II.
5. If the region bounded by the most exterior of the minimal curves $L$ were not absolutely convex, then there would be an arc lying outside $L$, joining two points of $L$ and having the same length as the corresponding piece of $L$. For reasons analogous to those presented above this arc would pass inside $G$ and even inside $V$. Replacement of a piece of $L$ by this arc gives a loop of the same length, enveloping the curve $L$. This contradicts the choice of $L$ as the most exterior of the minimal curves.

6. Each of the shortest curves in the neighborhood of one of its points not lying in $\gamma$ is a shortest arc on a small portion of itself, for otherwise it could be shortened. It preserves this property also at a point where it overlies the boundary $\gamma$, given only that the curve $\gamma$ itself is a shortest arc in neighborhood of this point. Therefore each of the minimal curves in question is a geodesic polygon (or closed geodesic). The vertices of this polygon can lie only at the vertices of the polygon $\gamma$.

Remark. In our proof the existence of a shortest loop was guaranteed by the local character of the construction; the proof was carried out in a fixed region $G$. This was not a superfluous complication, since it is easy to see, for example, that on an onion-shaped surface with a stem narrowing to infinity, a closed loop may go to infinity while constantly contracting.

The fact that the shortest loop does not lie along the boundary of the region in question was guaranteed by the fact that the enveloping figure has a sufficiently small perimeter. Up to now we have assumed that such a figure exists and we shall prove its existence below.

5. Neighborhood with a small perimeter.

Lemma 5. For any neighborhood $V$ of the point $O$ and any $\varepsilon > 0$ there exists a neighborhood of the point $O$ lying in $V$ and bounded by a simple closed polygon $\gamma$ with perimeter $p < \varepsilon$ such that all its vertices lie at points through which there pass shortest arcs.

In this section we have up to now used only the intrinsic character of the metric and the two-dimensionality of the space. In proving Lemma 5 it is necessary to use the condition of boundedness of the curvature, since without it the lemma is not true. For example, take a space which is the surface of a right circular cylinder extending to infinity in one direction, all the points of the base being identified and regarded as one point. This example has already been considered in subsection 2 of
Chapter II. For the point which is the base of the cylinder there evidently exists no neighborhood homeomorphic to the disc with a perimeter smaller than the length of the circumference on which the cylinder was constructed.

We turn to the proof of Lemma 5.

1. From Lemma 3, the point 0 can be encircled in an arbitrarily small neighborhood by a simple closed polygon \( \Gamma \) with vertices at points through which there pass shortest arcs. Suppose that \( d \) is the maximum and \( \rho > 0 \) the minimum of the distances of points of \( \Gamma \) from 0. Introducing superfluous vertices on the polygon \( \Gamma \), if necessary, we may suppose that each of the links of this polygon is not longer than \( \rho \).

Join the point 0 of the shortest arc \( L_1 \) to any vertex \( X \) of the polygon \( \Gamma \), avoiding any superfluous intersections with the links of \( \Gamma \). If \( L_1 \) has a single point in common with \( \Gamma \), which will then be the endpoint \( X \), we denote the polygonal line \( \Gamma \) by \( \Gamma_1 \) and pass to the following vertex of \( \Gamma \). If the common portion of the curves \( L_1 \) and \( \Gamma \) consists of more than one point, then we take the point \( A \) on \( L_1 \) common with \( \Gamma \) which is closest to 0 (Figure 18). Moving away from \( A \) on one of the sides along \( \Gamma \), we find on \( L_1 \) another point \( B \) in common with \( \Gamma \) such that the arc \( AB \) of the polygon \( \Gamma \) and the segment \( AB \) of the shortest arc \( L_1 \) form a simple polygon \( \Gamma_1 \) which encloses the point 0. Evidently all the points of the polygon \( \Gamma \) are separated from 0 by a distance not larger than \( d \) and not less than \( \rho \). In what follows we shall consider only the polygon \( \Gamma_1 \) and the segment \( OA \) of the shortest arc \( L_1 \).

The number of vertices of the original polygon \( \Gamma \) remaining on \( \Gamma_1 \) and not already joined with 0 by shortest arcs, has been reduced in comparison with \( \Gamma \) by at least unity. We choose one such vertex and join it to 0 by a shortest arc, not creating any superfluous intersections with \( OA \) and the links of \( \Gamma_1 \). As before, we pass to a simple polygon \( \Gamma_2 \) enclosing 0 on which the number of vertices not joined to 0 will be still smaller.

After a finite number of such steps we obtain a simple closed polygon \( \Gamma' \) enclosing 0. Its points are distant from 0 by a distance not less than \( \rho \) nor greater than \( d \). Each vertex of \( \Gamma' \) is joined to 0 by a shortest arc. These shortest arcs do not cross one another, coinciding possibly
only on the initial segments contiguous to \( O \). Along with the links of the polygon \( \Gamma' \) they form a system of nonoverlapping reduced triangles with vertices at \( O \) (Figure 19).

These shortest arc decompose a neighborhood of the point \( O \) into a finite number of sectors. It suffices to consider only those which are nondegenerate. They already form a decomposition of a sufficiently small neighborhood of the point \( O \). Each of these sectors lies in a triangle homeomorphic to a disc, in which the side opposite to \( O \) is shorter than \( \rho \), since it is a link or a portion of a link of the original polygon \( \Gamma \).

![Figure 19.](image)

2. Suppose that at least one of the sectors contiguous to \( O \) is not convex relative to the boundary. Then close to \( O \) on its sides there are points \( X \) and \( Y \) for which the shortest of the arcs joining them outside the sector will not coincide with the boundary of that sector, i.e. it will enclose the point \( O \) and will be shorter than \( XO + OY \). This arc (it will be a geodesic) and the portion of the perimeter of the triangle to which the sector in question belongs form a simple closed polygon encircling \( O \) (Figure 20). Its perimeter will be \textit{a fortiori} smaller than \( 5d \). Inasmuch as \( d \) may be taken arbitrarily small, this neighborhood will satisfy the requirements of the lemma.

It thus remains for us to consider the case when all the sectors of the subdivision thus obtained of the neighborhood of the point \( O \) are convex relative to the boundary.

3. We suppose that all the sectors are convex relative to the boundary and that there are none among them which are extended (i.e. which have sides extending one another through the point \( O \) and forming one shortest arc). Then we take on each shortest arc a point close to \( O \). We join these points by a sequence of shortest arcs in each sector. We obtain a polygon encircling \( O \) and having a small perimeter. If the sides issuing from one of its vertices coincide on an initial segment, we reject that
segment. We obtain the simple polygon required by the lemma.

4. We suppose finally that the sectors are convex relative to the boundary, but that among them there are some which are extended. Suppose that $C$ is such a sector. In exactly the same way as in the first part of the proof, in the sector $C$ close to $O$ we can draw a simple polygonal curve joining the sides of the sector and excising from $C$ a region homeomorphic to the disc, such that all the vertices of the curve will be at points through which there pass shortest arcs, and these vertices will be joined to $O$ by shortest arcs lying in $C$ without superfluous intersections with one another and with the sides of the sector $C$, and the links of the polygon will be shorter than any of these shortest arcs.

If these shortest arcs divide $C$ into sectors that are not extended, our problem will be completely solved.

Suppose however that among the sectors into which $D$ is decomposed, there again appears at least one which is extended. In our construction it lies in some triangle $T$ homeomorphic to a disc. Our immediate object is to prove that in this case there is a simple triangle with a significant positive excess lying in the sector $C$ close to $O$. We cannot assert that this will be the triangle $T$ itself ($T$ may be not even convex), and therefore we must turn from $T$ to a somewhat different figure.

The perimeter of $T$ may be taken to be so small so that from Lemma 4 there is a shortest loop among the loops close to $O$ enclosing $T$. Because of the convexity relative to the boundary of $C$ this loop lies in $C$ and therefore passes through $O$. By Lemma 4 this loop is a geodesic polygon. It bounds a figure $T'$ which is convex relative to the boundary. Besides $O$, the loop may either pass through both of the other vertices of the triangle $T$, or through one of them, or completely miss them and have the form of a geodesic loop with one vertex $O$. We consider in detail each of these three cases.

5. Case 1. If the loop passes through all the three vertices of $T$, then $T'$ coincides with $T$. This case is of particular interest. By hypothesis, the angle of this triangle adjacent to $O$ is $\alpha = \pi$. But from the fact that the side opposite to $O$ is shorter than the two other sides of $T$, it follows that in a plane triangle with sides of the same length we have for the corresponding angle $\alpha_0 < \pi/3$.

From Theorem 5 of Chapter II we have:

$$\alpha - \alpha_0 \leq \nu_{\phi}(T),$$
and therefore
\[ \nu_0^+(T) > \frac{2}{3}\pi. \]

Using the convexity relative to the boundary of \( T = T' \), we therefore conclude that one may single out in \( T \) a simple triangle with an excess larger than \( 2\pi/3 \). It in fact suffices to cut out a reduced triangle with an excess almost equal to \( \nu_0^+ \) and to discard its exterior tails.

6. Case 2. Suppose that, except for \( O \), the vertex \( A \) is the only vertex of the triangle \( T \) through which the loop \( T' \) passes. Since it is the interior one of the shortest loops, it consists of the side \( OA \) and the geodesic \( L \) (Figure 21).

We shall move from \( O \) along \( L \) so long as the part traversed remains a shortest arc. If we arrive in this manner at \( A \), then \( T' \) is seen to be two-sided. Choosing on its side an additional vertex \( X \), we convert it into a simple triangle with an excess not less than \( \pi \).

If now, moving along \( L \) from \( O \), we arrive at a last point \( X \) preceding \( A \) for which the traversed part of \( L \) is still a shortest arc, then the points of the portion \( AX \) must be joined to \( O \) by shortest arcs passing along the other side of \( T \) or inside it. The point \( X \) must itself have the same property. If two shortest arcs \( OX \) enclose \( T \), they form a two-sided figure which may be considered to be a simple triangle with an excess not less than \( \pi \). If now the second shortest arc \( OX \) passes inside \( T \), then the piece \( AX \) of the curve \( L \) is itself a shortest arc. Otherwise the loop \( T' \) would not be the shortest or most interior. In this case \( T' \) is the simple triangle \( OAX \) with an excess not less than \( \pi \).

7. Case 3 leads to the same result as in Case 2. It is sufficient to replace \( A \) by the point \( Y \) obtained in the same way as \( X \) (Figure 22).

8. We return to Lemma 5. Let us consider the case when the neighborhood of the point \( O \) was divided into sectors which were convex rela-
tive to the boundary. We then undertook to divide each of the extended sectors into nonextended sectors. This could turn out unsuccessful for not more than \( \left\lceil \frac{3C(G)}{2\pi} \right\rceil \) pieces of "plane" sectors, since in each of these sectors we can find a simple triangle with an excess larger than \( 2\pi/3 \), and the sum of the excesses of these triangles, by the hypothesis of boundedness of the curvature, does not exceed some quantity \( C(G) \).

9. Now we can easily obtain the neighborhood required in Lemma 5. We decompose a portion of the sectors into nonextended sectors. Suppose that \( m \) is the total number of nonextended sectors thus obtained. Laying off on their sides from \( O \) segments of length \( \varepsilon/4m \) and joining the points thus obtained in each sector by shortest arcs, we obtain a polygonal arc with a total length not greater than \( \varepsilon/2 \). We complete it to a closed arc by going around the outer contour of the triangle to which the remaining extended sectors belong (Figure 23). This still leaves a length not larger than \( \left( \frac{3C(G)}{2\pi} \right)3d \). Because of the smallness of \( d \) it may also be made less than \( \varepsilon/2 \).

We thus obtain a neighborhood of the point \( O \) homeomorphic to the disc, which is bounded by a simple polygonal arc \( \gamma \) with length less than \( \varepsilon \), with the vertices of the polygonal arc lying at points through which there pass shortest arcs.

Lemma 5 is proved.

6. Absolutely convex neighborhood. Applying Lemma 4 to the curve \( \gamma \) obtained in Lemma 5, and choosing the most exterior of the shortest loops enclosing \( \gamma \) near \( O \), we obtain the following result.

**Theorem 1.** For each point there exists an arbitrarily small, absolutely convex neighborhood \( P \), bounded by a polygonal arc of arbitrarily small length, while each vertex of this polygonal curve lies at a point through which a shortest arc may be passed.

2. Triangulation. In this section we establish the possibility of decomposition of polygons into arbitrarily small triangles. We recall that a polygon is a connected compact set bounded by a finite number of simple closed polygonal curves (subsection 15 of Chapter I). Among their number we include figures having no boundary at all. For example in the space represented by the intrinsic geometry of the surface of the torus, the entire surface of the torus also is regarded as a polygon.
The precise formulation of the basic result of this section is given in Theorem 3.

7. Covering by triangles.

Theorem 2. Every compact set admits a covering by a finite system of arbitrarily small nonoverlapping simple triangles.

We may suppose in addition that all the vertices of these triangles lie at points through which there pass shortest arcs.

For the proof of Theorem 2 we establish one auxiliary result.

Lemma 6. If the absolutely convex polygon \( P \) intersects nonoverlapping polygons \( Q_1, \ldots, Q_n \), which are convex relative to their boundaries, then the entire figure \( P + \sum Q_i \) may be decomposed into a finite number of nonoverlapping polygons convex relative to their boundaries and homeomorphic to the disc.

Proof. The case \( P \subset \sum Q_i \) is trivial. For the same reason we may exclude from consideration all \( Q_i \subset P \) and all \( Q_i \) having no points in common with \( P \). Observe that since no one shortest arc with ends in \( P \) leaves \( P \), each side of any of the polygons \( Q_i \) can intersect the boundary of \( P \) in an essential way not more than twice. Therefore the number of intersections of the boundaries of \( P \) and \( Q \) is finite. (Excluding overlaps and tangencies of the boundary of \( Q_i \) to the boundary of \( P \) from inside.)

Consider the pair \( P, Q_1 \). By hypothesis, \( Q_1 \) intersects \( P \) but is not contained in \( P \). The part of the boundary of \( Q_1 \) passing outside \( P \) consists of simple arcs \( \overline{A_1B_1}, \overline{A_2B_2}, \ldots \) (Figure 24). The number of these arcs is finite. Along with the pieces of the boundary \( P \) they form regions homeomorphic to the disc, no one of which falls inside another. (If this latter property did not hold, the intersection of \( P \) with \( Q_1 \) would be disconnected, which cannot happen since that intersection is convex in view of the absolute convexity of \( P \) and the convexity of \( Q_1 \).)

We join the ends of the arc \( \overline{A_1B_1} \) by the shortest arc \( \overline{A_1B_1} \) in \( Q_1 \). This shortest arc also passes inside \( P \). Analogously we lay out the shortest arcs \( \overline{A_2B_2}, \overline{A_3B_3} \) and so forth. We may suppose that these shortest arcs are laid out without superfluous intersections with the sides of the polygons containing them and with one another. Therefore they either have no common points one with another, or else they only touch one another
at a point or along some segment. Thus we excise by shortest arcs the parts of $Q_i$ protruding from $P$.

Next we will start excising the parts of $Q_2$ by shortest arcs. Since the $Q_i$ are all nonoverlapping, we do not encounter the interior parts of any of the $Q_i$, in passing shortest arcs in $Q_2$. Thus it is possible for us to carry to completion the process of all the parts of $Q_1, \ldots, Q_n$ which protrude from $P$.

Each excised part is a polygon convex relative to its boundary. The remainder of the polygon $P$ itself after all the excisions turns out also to be divided into a finite number of polygons convex relative to their boundaries. It is easy to show this by induction on the number of shortest arcs drawn in $P$.

Thus Lemma 6 is proved.

Now we turn to the proof of Theorem 2. By Theorem 1, each point can be encircled by an arbitrarily small absolutely convex polygon with vertices at points through which there pass shortest arcs. In view of the compactness of the set we may select a finite subcollection from these coverings. Suppose that part of these polygons are already decomposed into nonoverlapping polygons convex relative to the boundary. Let us add another which is absolutely convex. By Lemma 6, the collection with this adjunction is again decomposable into polygons convex relative to their boundaries. Thus, adding one at a time all the polygons of the finite covering in question, we decompose that covering into polygons convex relative to the boundary with vertices at points through which there pass shortest arcs.

Each polygon thus obtained may be decomposed into reduced triangles convex relative to the boundary by diagonals issuing from one vertex. Discarding if necessary the exterior tails from these triangles, we obtain the required covering by simple triangles.

Theorem 2 is proved.

We observe that the triangles obtained above cannot adjoin one another along entire sides.

**Lemma 7.** Under the hypotheses of Theorem 2 it is possible to ensure that for each of the triangles no side is equal to the sum of two others.

**Proof.** Suppose that one of the simple triangles is a two-sided figure $(AB)$. Mark of the midpoints $C$ and $C'$ of its sides. If $CC' = AB$, then the path $CBC'$ is a shortest arc. Shifting somewhat the points $C$ and $C'$
on the side of $B$, we convert $(AB)$ into the required triangle $ACC'$.

Suppose that on one or both endpieces $CC'$ coincides with pieces of the side $(AB)$ which depart from them at the points $D$ and $E$ (Figure 25). For definiteness we suppose that $CD \geq C'E$. In this case either $ED$ decomposes $(AB)$ into two triangles of the sort required, or $CD = DE + EC'$, and we obtain two new two-sided figures: $(AD)$ and $(BE)$. On the contour of the two-sided figure $(AD)$ the piece $CDE$ is a shortest arc. Therefore, shifting $C$ towards $D$, we convert $(AD)$ into the required triangle $ACE$, We apply the same process to the two-sided figure $(BE)$ as to the original two-sided figure $(AB)$.

This process cannot be continued indefinitely, otherwise we would obtain infinitely many nonoverlapping two-sided figures of the type $(AD)$. For each of them the angle at the vertex $D$ is equal to $\pi$. Therefore they may be considered as simple triangles with an excess greater than or equal to $\pi$. Thus if there were infinitely many of them, they would close down toward some point and the condition of boundedness of the curvature would be violated in its neighborhood.

Lemma 7 is proved.

8. Decomposition into triangles.

**Theorem 3.** Every polygon $M$ without boundary or with a boundary, the sectors at the vertices of which are convex relative to the boundary or are made up of sectors convex relative to the boundary, may be decomposed (triangulated) into arbitrarily small simple triangles. (We do not require that they should adjoin one another along entire sides.) We may also arrange matters so that the following additional requirements are satisfied: 1) all the vertices of the triangles, with the exception of the vertices of the original polygon $M$, lie at points through which there pass shortest arcs; 2) in each triangle the sum of any two of its sides is greater than the third; 3) in the net of curves of the decomposition there are included the initial portions adjacent to the vertices of the polygon $M$ of an arbitrarily given finite system of shortest arcs, dividing the sectors at the vertices of $M$.

If the additional conditions are discarded, then, as in Theorem 1, one may assert the possibility of covering $M$ by a finite number of nonover-
lapping triangles, only this time in such a way that they do not extend outside the limits of \( M \). The case of the absence of a boundary for \( M \) therefore does not present anything new, and we may carry out the proof of Theorem 3 on the assumption that \( M \) has a boundary.

First we encircle each vertex \( A_i \) of the polygon \( M \) by an absolutely convex polygon \( P_i' \). We may take \( P_i' \) to be so small that the following conditions are also satisfied. 1. The \( P_i' \) do not intersect one another. 2. The portion of \( P_i \) adjacent to \( A_i \) (cross-hatched in Figure 26) of the intersection of \( P_i' \) with \( M \) is a polygon convex relative to the boundary or else consists of such polygons. This may be arranged since the sector of \( M \) is convex relative to the boundary or consists of pieces which are convex relative to the boundary. 3. The given shortest arcs issuing from \( A_i \) which divided the sector of the polygon \( M \) leave the limits of \( P_i' \). 4. The vertices of \( P_i' \) distinct from \( A_i \) lie at points through which there pass shortest arcs.

On each side \( l_i \) of the polygon \( M \) we discard terminal segments lying entirely inside \( P_i' \) and \( P_i'_{i+1} \). We obtain a closed segment \( \overline{l}_i \) (Figure 26). Each of its points \( X \) may be enclosed in an arbitrarily small absolutely convex polygon. The interior part of this polygon along with the point \( X \) of the segment \( \overline{l}_i \) covers an open segment \( U \) adjacent to \( X \). From the covering of \( \overline{l}_i \) by these segments we may extract a finite subcovering. We thus obtain a covering of all the \( \overline{l}_i \) by a finite number of polygons \( Q_i' \), which may be regarded as so small that they do not touch the vertices \( A_i \), and those of the polygons \( Q_i' \) which cover one side of \( M \) do not touch the polygons which cover another one of its sides. From each polygon \( Q_i' \) we preserve only that part \( Q_i \) which belongs to the intersection of \( Q_i' \) and \( M \), is homeomorphic to the closed disc, and includes the segment \( U_i \) (the part \( Q_i \) is cross-hatched in Figure 27).

Finally, the interior part of \( M \), still not covered by the interior pieces of all the \( P_i \) and \( Q_i \), will be
We note that although the polygons $P_t$ and $Q_y$ are not necessarily fully convex, they are convex relative to the boundary and contain every shortest arc joining two of its points, given only that this shortest arc lies in $M$. Therefore the entire system of polygons $\sum_i P_i + \sum_j Q_j + \sum_k R_k$ may be decomposed into a finite number of nonoverlapping polygons which are convex relative to the boundary, proceeding in exactly the same way as in the proof of Lemma 6. We need only to start with the polygons $P_i$ and then add the rest one by one and carry out the subdivision. Moreover, the pieces adjacent to the $A_t$ can be regarded as subdivided by the given shortest arcs which issue from $A_t$. It remains to lay off diagonals in the polygons, and we obtain a subdivision into simple triangles.

The additional conditions 1) and 2) are obtained as in Theorem 2 and Lemma 7. Theorem 3 is completely proved.


9. **A consequence of Euler’s theorem.** Consider a polygon $P$ divided into simple triangles (which are not required to be adjacent to one another along entire sides). The decomposition will be called a triangulation; the vertices of the triangles are vertices of the triangulation, and the segments of the sides of the triangles from one vertex of the triangulation to another are the edges in the net of the triangulation. We suppose that all the vertices of $P$ are counted among the vertices of the triangulation.

We introduce the following notation.

- $n$: the number of vertices of $P$,
- $m$: the number of vertices of the triangulation on the sides of $P$,
- $e$: the number of vertices of the triangulation inside $P$,
- $f$: the number of triangles of the triangulation,
- $k$: the number of edges in the net of the triangulation,
- $\nu$: the number of cases when a side of a triangle passes through a vertex of the triangulation (if several sides of several triangles pass through one vertex, these are counted in $\nu$ the corresponding number of times),
- $\chi$: the Euler characteristic of the polygon $P$.

From Euler’s theorem we have:

$$(n + m + e) - k + f = \chi.$$
On the other hand,

\[ 2k = 3f + m + n + \nu, \]

so that

\[ f + \nu = 2e + n + m - 2\chi. \]

Suppose that \( \alpha \) denotes all possible angles adjacent to the interior vertices of the triangulation, \( \beta \) the angles adjacent to the sides of \( P \), \( \xi \) the angles of the triangles at the vertices of \( P \), and finally \( \phi \) the auxiliary angles equal to \( \pi \), introduced in all \( \nu \) cases when the side of the triangle passes through the vertex of the triangulation.

Then for the sum of the excesses of all triangles of the triangulation we will have

\[
\sum \delta = \sum \alpha + \sum \beta + \sum \xi + \sum \phi - f\pi - (f + \nu)\pi
\]

\[ = \sum \alpha + \sum \beta + \sum \xi + \sum \phi - \pi(2e + n + m - 2\chi) \]

\[ = \left[ \sum \alpha + \sum \phi - 2\pi e \right] + \left[ \sum \beta - m\pi \right] + \left[ \sum \xi - (n - 2\chi)\pi \right]. \]

We may formulate the above result in the following way.

**Lemma 8.** For a triangulation of a polygon \( P \) into simple triangles the sum of the excesses of the angles of these triangles may be decomposed into three terms:

\[ \sum \delta = \text{I} + \text{II} + \text{III}, \]

where the first term is the excess over \( 2\pi \) of the sum of the angles adjacent to a vertex, the summation being taken over all the interior vertices of the triangulation:

(1) \[ \text{I} = \sum_{i=1}^{\nu} \left( \sum_i \alpha + \sum_i \phi - 2\pi \right); \]

the second term the excess over \( \pi \) of the sum of the angles, the summation being taken over all the vertices of the triangulation, lying on the sides of \( P \):

(2) \[ \text{II} = \sum_{j=1}^{m} \left( \sum_j \beta - \pi \right); \]

and finally, the third term is:

(3) \[ \text{III} = \sum \xi - (n - 2\chi)\pi. \]

In the case of a polygon \( P \) homeomorphic to a disc, \( \chi = 1 \) and the third term is the excess of the sum of the angles adjacent to the vertices of \( P \) over the sum of the angles of a plane \( n \)-gon:

(4) \[ \text{III} = \sum \xi - (n - 2)\pi. \]
All the remaining results of this section are based on Lemma 8.

10. Nonlocalness of the condition of boundedness of the curvature.

**Lemma 9.** Every geodesic triangle $T$ homeomorphic to a disc, whose sectors at the vertices are convex relative to the boundary (or made up of pieces convex relative to the boundary) may be decomposed into arbitrarily fine simple triangles $t$ such that the inequality

$$\sum \delta(t) \geq \delta(T)$$

holds. Moreover, for any $\varepsilon > 0$ we may arrange matters so that the inequality

$$\sum \delta(t) \geq \bar{\delta}(T) - \varepsilon,$$

holds, where $\bar{\delta}(T)$ is the excess over $\pi$ of the sum of the angles of the sectors at the vertices of $T$.

Indeed, from Theorem 3 we can triangulate $T$ into fine triangles $t$ such that the vertices of the triangulation other than the vertices of $T$ lie at points through which there pass shortest arcs. Then in the sum

$$\sum \delta(t) = I + II + III$$

all the terms of the sum $I(1)$ will be nonnegative in view of Theorem 11 of Chapter II. The terms of the sum $II(2)$ will be nonnegative according to Theorem 2 of Chapter II on adjacent angles. Finally, the sum $III(4)$ is not less than the excess $\delta(T)$. Therefore

$$\sum \delta(t) \geq III \geq \delta(T).$$

If among the curves of the triangulation the shortest arcs issuing from the vertices of the triangle $T$ were included, and these shortest arcs divided each of its sectors into pieces with a sum of angles close to the value of the angle of the sector, then

$$\sum \delta(t) \geq III \geq \bar{\delta}(T) - \varepsilon.$$  \(\text{(5)}\)

**Theorem 4.** For any finite system of simple nonoverlapping triangles $T$ lying in a compact set $M$, the sum $\sum \delta(T)$ is bounded by a finite number depending only on $M$. Moreover, $\sum \bar{\delta}(T)$ is bounded by the same number.

**Proof.** By the hypothesis of boundedness of the curvature each point $X \in M$ has a neighborhood $G(X)$, homeomorphic to the open disc, in which the sum of the excesses of simple nonoverlapping triangles is bounded above by some number $C(G)$. Select a finite covering of $M$ by regions $G_i = G(X_i)$, $i = 1, \ldots, n$.

---

3 We still have not established the finiteness of the angles of the sectors. Under the hypothesis $\bar{\delta}(T) = \infty$ one may think of an arbitrarily large number instead of the expression $\bar{\delta}(T) - \varepsilon$. 
Suppose that a system of \( m \) nonoverlapping simple triangles lies in \( M \). Reject those for which the excess is nonpositive. We decompose the remaining ones, using Lemma 9, into triangles \( t \) such that the sum of the excess does not decrease, and such that each of the triangles \( t \) is so small that it is entirely contained in one of the regions \( G(X_i) \). Then

\[
\sum \delta(T) \leq \sum \delta(t) \leq \sum_{i=1}^{n} C(G_i)
\]

and the first assertion of the theorem is proved. Moreover, we may assume that

\[
\sum \delta(T) \leq \sum \delta(t) + m \varepsilon,
\]

so that in view of the arbitrariness of \( \varepsilon > 0 \) the second assertion of Theorem 4 holds.

The least upper bound of the sum of the positive excesses of nonoverlapping simple triangles contained in the set \( M \) will be from now on denoted by \( C(M) \).

11. Preparatory estimates.

**Lemma 10.** In a geodesic \( n \)-gon \( P \) homeomorphic to a disc, in which the sectors at the vertices are convex relative to the boundary or made up of pieces convex relative to the boundary, the sum of the angles of the sectors will exceed the sum of the angles of a plane \( n \)-gon by not more than \( C(P) \).

**Proof.** Suppose that \( \xi \) are the angles of the sectors at the vertices of \( P \). We subdivide these sectors by shortest arcs so that for the angles \( \xi' \) between the sides of the sectors convex relative to the boundary thus obtained the inequality

\[
\sum \xi' > \sum \xi - \varepsilon
\]

will hold. Then we triangulate \( P \) so that the vertices of the triangulation will lie at points through which there pass shortest arcs, and the initial segments of the shortest arcs drawn above enter into the structure of curves to the triangulation. Then we have:

\[
C(P) \geq \sum \delta = I + II + III \geq III
\]

\[
\geq \sum \xi' - (n - 2)\pi
\]

\[
> \sum \xi - (n - 2)\pi - \varepsilon.
\]

From the arbitrariness of \( \varepsilon > 0 \) we therefore have

\[
\sum \xi - (n - 2)\pi \leq C(P).
\]
For $P$ not homeomorphic to a disc we obtain analogously
\[ \sum \xi - (n - 2\chi)\pi \leq C(P). \]

**Lemma 11.** If the neighborhood of the point $A$ is divided into sectors convex relative to the boundary, and $\alpha_i$ are the angles between their sides, then
\[ 2\pi - \sum \alpha_i \leq C(A), \]
where $C(A)$ is the upper limit of the quantities $C(P)$ for all possible polygons, homeomorphic to the disc, which contain $A$ in the interior and converge to $A$.

For the proof we join successively the points $X_i, X_{i+1}$ on the shortest arcs issuing from $A$ and dividing the neighborhood of the point $A$, by the leftmost shortest arcs $X_iX_{i+1}$. If at least one of the shortest arcs $X_iX_{i+1}$ passes through $A$, then $2\pi - \sum \alpha_i \leq 0$ and the assertion of the lemma is trivial. Otherwise, we construct a polygon $P'$ enclosing $A$ (Figure 28). It is homeomorphic to the disc or still has exterior tails. The polygon $P'$ decomposes into reduced triangles convex relative to the boundary, $t_i = AX_iX_{i+1}$. Although they are not homeomorphic to the disc, we evidently are allowed to apply Lemma 8:
\[ \sum \delta(t_i) = I + II + III. \]

In this case the sum II disappears, so that
\[ -I = III - \sum \delta(t_i), \]
or in other words
\[ 2\pi - \sum \alpha_i = \left[ \sum \xi - (n - 2)\pi \right] - \sum \delta(t_i) \leq \left[ \sum \xi - (n - 2)\pi \right] - \sum \delta(t_i). \]

By Lemma 10, the first of the terms on the right does not exceed the quantity $\sum \xi - (n - 2)\pi$ for a polygon $P$ obtained from $P'$ by rejecting the exterior tails. By Lemma 11 of Chapter II all the terms of the second term tend to zero as $(X_i, \cdots, X_n) \to A$. Therefore
\[ 2\pi - \sum \alpha_i \leq \lim \sup C(P) \leq C(A). \]

The following lemma results immediately from Lemma 11.
Lemma 12. For a finite system of points $A_i$ lying, along with neighborhoods which admit decomposition into sectors convex relative to the boundary, in a compact set $M$
\[ \sum_{A_i}(2\pi - \sum_i \alpha) \leq C(M), \]
where $\sum_i \alpha$ is the sum of the angles between the sides of the sectors convex relative to the boundary into which the neighborhood of the point $A_i$ is decomposed.

Lemma 13. For any triangulation of a polygon $P$, homeomorphic to the disc, into simple triangles $T$, the sums of the excess of these triangles are bounded by numbers depending only on $P$:
\[ \sum \delta^+(T) \leq C(P), \]
\[ \sum \delta^-(T) \leq 2C(P) + (n - 2)\pi, \]
where $\delta^+$ and $\delta^-$ are the positive excess and the absolute value of the negative excess, and $n$ is the number of vertices of $P$.

The first assertion follows from the definition of $C(P)$, the finiteness of $C(P)$ being asserted by Theorem 4. Therefore it remains to prove the estimate (8). We have
\[ \sum \delta^-(T) = -\sum \delta(T) + \sum \delta^+(T) \leq -I - II - III + C(P). \]
But
\[ -II \leq 0, \]
\[ -I = \sum (2\pi - \sum \alpha) \leq C(P), \]
\[ -III = -\sum \xi + (n - 2)\pi \leq (n - 2)\pi. \]
Estimate (8) follows.

In case of a polygon of different structure we obtain instead of (8)
\[ \sum \delta^-(T) \leq 2C(P) + (n - 2\chi)\pi. \]

Remark. If in some triangles $T$ of the indicated triangulation one excises triangles $T'$ by joining two points on the lateral sides by leftmost shortest arcs, then what remains of the triangles $T$ will be convex relative to the boundary. They may be decomposed into triangles convex relative to the boundary, which along with the triangles $T'$ constitute a new triangulation of the polygon $P$. Although some of the triangles entering into this new triangulation (the triangles $T'$) may turn out not to be homeomorphic to the disc, they will be reduced triangles, convex relative to the boundary and without interior tails. One may retrace the steps and see that for this sort of triangulation the assertion of Lemma
13. Estimates of the curvature and rotations in the constructed development. The estimates presented in this section are very rough. But what is important to us now is only their qualitative derivation. In Chapter V several of these estimates will be substantially sharpened.

**Lemma 14.** The positive part of the curvature of \( P_0 \) does not exceed \( 4C(P) \):
\[ \omega^+(P_0) \leq 4C(P). \]

**Proof.** Write down the evident equation

\[ \omega^+(P_0) = \sum' \left( 2\pi - \sum'' \alpha \right) = \sum' \left( 2\pi - \sum'' \alpha \right) + \sum' \sum'' (\alpha - \alpha_0). \]

Here \( \alpha \) and \( \alpha_0 \) are corresponding angles in the triangles \( T \) and \( T_0 \), the sum \( \sum' \) is extended over all the vertices of the development for which \( \omega^+ > 0 \) and the sum \( \sum'' \) over the angles adjacent to the corresponding concrete vertex.

From Theorem 5 of Chapter II on the angles of a triangle and from the condition of boundedness of the curvature, the rough estimate

\[ \sum (\alpha - \alpha_0) \leq 3C(P) \]

follows evidently for the sum \( \sum \) extended over any group of angles of the triangles of the triangulation. Moreover, from Lemma 12,

\[ \sum' \left( 2\pi - \sum'' \alpha \right) \leq C(P). \]

Together these prove (9).

In the original triangle for each interior point \( A \) on a side of \( T \) the sum of the angles between the edges of the triangulation adjacent to \( A \) and on one side of \( T \) will not be less than \( \pi \), as follows from the theorem on adjacent angles. Therefore the phenomenon mentioned above, namely, that at the point of the development \( P_0 \) corresponding to \( A \) there is a positive exterior rotation on a side of \( T_0 \) (Figure 29), can be caused only by the contraction of angles adjacent to \( A \) outside of \( T \). Along with the estimate (10) this leads to the following assertion.

**Lemma 15.** *The sum of the positive parts of the exterior rotations \( \tau_0^+ \) of the sides of the triangles \( T_0 \) in the structure of the development \( P_0 \) and the positive rotations on the boundary of \( P_0 \) (not taking into account the rotations at points corresponding to the vertices of \( P \) itself) does not exceed \( 3C(P) \):*

\[ \sum \tau_0^+ \leq 3C(P). \]

Indeed,

\[ \sum \tau_0^+ = \sum (\pi - \sum' \alpha_0) = \sum (\pi - \sum' \alpha) + \sum \sum' (\alpha - \alpha_0) \leq 0 + 3C(P). \]

Here the sum \( \sum \) was extended over those points for which there is one of the positive rotations \( \tau_0^+ \), and \( \sum' \) over the angles of the triangles \( T, T_0 \) in \( P \) or \( P_0 \), adjacent to this point from one side.

If to the subdivision of \( P_0 \) into triangles \( T_0 \) we apply the Euler theorem, i.e., if we repeat the reasoning that led to Theorem 8, then we obtain
0 = [ω−(P₀) − ω⁺(P₀)] + [τ−(Γ₀) − τ⁺(Γ₀)] + \left[\sum \xi_0 - (n - 2\chi)\pi\right],

where \(\tau^+(\Gamma_0) - \tau^-(\Gamma_0)\) is the interior rotation of the boundary curves of \(P₀\), excluding their rotation at points corresponding to the vertices of \(P₀\), and \(\xi_0\) are the angles of the sectors at the vertices of \(P₀\). It therefore follows that

\[(12) \quad \omega^-(P₀) + \tau^-(\Gamma_0) + \sum \xi_0 = \omega^+(P₀) + \tau^+(\Gamma_0) + (n - 2\chi)\pi.\]

Taking account of (9) and (11), we obtain the following result.

**Lemma 16.**

\[(13) \quad \omega^-(P₀) + \tau^-(\Gamma_0) + \sum \xi_0 \leq 7C(P) + (n - 2\chi)\pi.\]

From this inequality it follows that, separately, each term of the left side also does not exceed the expression on the right, so that in particular we have the following important result.

**Theorem 5.** The absolute curvature of the development \(P₀\) does not exceed some number which depends only on \(C(P)\), the number of vertices \(n\) and the Euler characteristic \(\chi\) of the polygon \(P\):

\[(14) \quad \omega^+(P₀) + \omega^-(P₀) \leq 11C(P) + (n - 2\chi)\pi.\]

5. **Some properties of a polyhedral metric.** As were the two preceding sections, this section is of an auxiliary nature. The results, presented in the form of theorems, are easily visualized and may have independent interest as well.

14. **Strengthening of the fundamental theorem on the angles of a triangle.**

**Lemma 17.** For every triangle homeomorphic to a disc the estimate

\[(15) \quad \bar{\alpha} - \alpha_0 \leq \omega^+,\]

holds, where \(\bar{\alpha}\) is the angle of a sector of that triangle, \(\alpha_0\) is the corresponding angle in the plane triangle with the same sides, and \(\omega^+\) is the positive part of the curvature of the interior region of the triangle.

The assertion of the lemma is not an easy consequence of Theorem 5 of the preceding chapter, since \(\bar{\alpha}\) is the angle of a sector which could be larger than the angle between the sides of the triangle.

Before proceeding to the proof of the lemma we note that the triangle may, without loss of generality, be regarded as convex, since it may be excised, and, by identifying along the edges with a second exemplar of the same triangle, it may be considered in the form of the resulting
two-dimensional manifold homeomorphic to a sphere. This does not change the quantities \( \check{a}, a_0, \omega^+ \) and it makes the triangle convex.

We turn to the proof of the lemma. From the Gauss-Bonnet theorem for polyhedral metrics, (13) of Chapter I, it follows that

\[
\pi - \check{a} + \pi - \check{\beta} + \pi - \check{\gamma} + \tau + \omega = 2\pi,
\]

where \( \tau = -\tau^- \) is the rotation of the sides of the triangle themselves, from a side turned toward the interior of the triangle, and \( \pi - \check{a}, \pi - \check{\beta}, \) and \( \pi - \check{\gamma} \) are the rotations at the vertices. Therefore

\[
(16) \quad \check{a} - a_0 + \check{\beta} - \beta_0 + \check{\gamma} - \gamma_0 = \omega^+ - \omega^- - \tau^- \\
\]

and

\[
(17) \quad \check{a} - a_0 + \beta - \beta_0 + \gamma - \gamma_0 \leq \omega^+.
\]

Here we have discarded nonpositive terms on the right, and, on the left side, we have passed from the sector angles \( \check{\beta}, \check{\gamma} \) to the angles \( \beta, \gamma \) themselves. This can only strengthen the inequality.

Further, inequality (15) follows from (17) in the same way as, in the proof of Theorem 5 of Chapter II, inequality (21) followed from (23).

Lemma 17 is proved. This lemma may be further strengthened.

**Theorem 6.** For every triangle homeomorphic to the disc, in the polyhedral metric the differences \( \check{a} - a_0, \check{\beta} - \beta_0, \check{\gamma} - \gamma_0 \) not only each separately, but in any sums by two, and also in the sum of all three, do not exceed \( \omega^+ \).

**Proof.** 1. For one difference the assertion was proved in Lemma 17. For the sum of the three differences the assertion follows trivially from (16). It remains to consider the most difficult case, namely to prove that

\[
(18) \quad \check{a} - a_0 + \beta - \beta_0 \leq \omega^+.
\]

The proof will be carried out by induction on the number of vertices with positive curvature included inside the triangle. If there are no such vertices, then \( \omega^+ = 0 \) and inequality (18) follows from the fact that separately \( \check{a} - a_0 \leq \omega^+ \) and \( \check{\beta} - \beta_0 \leq \omega^+ \). Suppose that the number of vertices with positive curvature inside the triangle is \( n \geq 1 \). We suppose that the theorem is proved for a smaller number of vertices.

2. From the assertion of Theorem 6 it follows that this assertion is valid not only for a triangle homeomorphic to the disc but also for any reduced triangle, excluding a triangle degenerating into one segment of a shortest arc.
Indeed, for a triangle consisting of tails alone, the theorem is trivial, since in this case $\alpha = \beta = \gamma = 0$. For a triangle having only interior tails, a simple cut along a segment of a shortest arc entering the triangle and then a pasting of two exemplars of such triangles leads to the case of a triangle homeomorphic to the disc. Finally, in the presence of an external tail it suffices to apply Theorem 6 to the remaining portion of the triangle, and then, now in the plane triangle, to adjoin the tail-like segment and carry out the further straightening of the “triangle.” In doing this the rectified angles of the plane “triangle” only grow, so that the validity of the theorem is preserved. This makes it possible to use the induction hypothesis for reduced triangles.

3. We continue the proof. As in Lemma 17, we take the triangle to be convex. We move a point along the side of the triangle opposite to $\alpha$ and all the time keep this point joined by a shortest arc to the vertex $A$ of the sector $\alpha$. Since by hypothesis there is a vertex with a positive curvature inside the triangle, through which no shortest arc may pass, then at some point $M$ it is possible to lay off two shortest arcs $AM$. The leftmost and rightmost of the shortest arcs $AM$ lying in the triangle form a two-gon, in which there is at least one vertex with positive curvature.

![Figure 30.](image)

The shortest arcs $AM$ divide the triangle into three parts as in Figure 30. In Figure 30, for simplicity, all three parts are depicted as figures homeomorphic to the disc. About this one may only assert that they have the form of reduced triangles, and, from the remark made above, we may apply the assertion of the lemma to each of them, given only that that triangle does not degenerate into a segment.

4. Note that the middle two-gon certainly does not degenerate into a segment. If the triangle $ACM$ degenerates into a segment, then $\gamma_0 \leq \pi$ and inequality (18) follows from the fact that

$$\alpha - \alpha_0 + \beta - \beta_0 + \gamma - \gamma_0 \leq \omega^+.$$

The possibility of degeneration into a segment of the triangle $ABM$ will be considered later on. At present we confine ourselves to the general
case when all three parts of the triangle do not reduce to a segment. The notations we shall use are adequately explained in Figures 30-33.

5. We consider on the plane the triangles $\triangle CAM$ and $\triangle AMB$ with the same sides. These triangles are taken to be adjacent along the side $AM$, as in Figure 31. In the plane quadrilateral thus obtained the angle $\alpha^0 = \alpha_1^0 + \alpha_2^0$ at the vertex cannot be reentrant, for otherwise we would have $CM + MB > CA + AB$. For the angle $\lambda^0 = \lambda_1^0 + \lambda_2^0$ at the vertex $M$ we may have two cases: $\lambda^0 \geq \pi$ and $\lambda^0 < \pi$. The first case is depicted in Figure 31, and the second in Figure 32.

Consider the case $\lambda^0 \geq \pi$. For the triangle $\triangle CAM$ we have:

$$\bar{\alpha}_1 - \alpha_1^0 \leq \omega^+_1;$$

for $\triangle ABM$ (in which there are fewer than $n$ vertices with positive curvature)

$$\alpha_2 - \alpha_2^0 + \beta - \beta^0 \leq \omega^+_2;$$

finally, for a two-gon ($\triangle AM$)

$$\bar{\alpha}_3 \leq \omega^+_3.$$

Combining these inequalities, we obtain

$$\bar{\alpha} - \alpha^0 + \beta - \beta^0 \leq \omega^+, \tag{19}$$

so that

$$(\alpha_0 - \alpha_0^0) + (\beta^0 - \beta_0),$$

where $\alpha_0$ and $\beta_0$ are angles in the plane triangle with sides $AB, AC, BC$.

Since in the rectification of the side $CM + MB$ in the quadrilateral of Figure 31 the angles $\alpha^0 = \alpha_1^0 + \alpha_2^0$ and $\beta^0$ can only grow, each of the quantities in brackets is nonpositive, so that (18) follows from inequality (19).

Now consider the case $\lambda^0 < \pi$. We have:

$$\bar{\alpha}_1 - \alpha_1^0 + \lambda_1 - \lambda_1^0 \leq \omega^+_1 \quad \text{(from $\triangle CAM$)},$$

$$\bar{\alpha}_2 - \alpha_2^0 + \beta - \beta^0 + \lambda_2 - \lambda_2^0 \leq \omega^+_2 \quad \text{(from $\triangle ABM$)},$$

$$CM + MB > CA + AB.$$
\[
\tilde{\alpha}_3 + \tilde{\lambda}_3 \leq \omega_3^+ \quad \text{(from the two-gon } AM) 
\]
so that in the sum
\[
\tilde{\alpha} - \alpha^0 + \tilde{\beta} - \beta^0 + \tilde{\lambda} - \lambda^0 \leq \omega^+ 
\]
or in other words
\[
(20) \quad \tilde{\alpha} - \alpha_0 + \tilde{\beta} - \beta_0 \leq \omega^+ + [(\alpha^0 + \beta^0 + \lambda^0) - (\alpha_0 + \beta_0 + \pi) + (\pi - \tilde{\lambda})].
\]

In the rectification of the side \(CM + MB\) in the plane quadrilateral of Figure 32, the angle at the vertex \(C\) decreases so that the sum of the remaining angles increases. In view of this, the first of the expressions in brackets on the right side of (20) is nonpositive. The third expression is nonpositive, since \(\tilde{\lambda}\) is the angle of the sector on one side from the shortest arc.

Therefore (18) follows from (20).

6. The remaining possibility of degeneracy of the triangle \(ABM\) into a segment \(AB + BM\) is treated analogously. We have (Figure 33):
\[
\tilde{\alpha}_1 - \alpha_0^0 + \tilde{\lambda}_1 - \lambda_1^0 \leq \omega_1^+, \quad \tilde{\alpha}_2 - 0 + \tilde{\beta} - \pi + \tilde{\lambda}_2 - 0 \leq \omega_2^+, 
\]
so that
\[
\tilde{\alpha} - \alpha_0 + \tilde{\beta} - \beta_0 \leq \omega^+ + (\pi - \tilde{\lambda}) + (\alpha^0 - \alpha_0) \leq \omega^+.
\]

Theorem 6 is completely proved.

Remark. 1) As we already noted in the course of the proof, the assertion of Theorem 6 remains valid for any reduced triangles other than those degenerating into segments. In this excluded case, for vertices lying inside the segment, the concept of a sector "on the side of the triangle" loses its meaning. For exterior sectors adjoining this vertex the difference \(\tilde{\alpha} - \alpha_0\) may be very large. It remains in this case to take, by definition, the "angle of the interior sector" to be the number \(\pi\). Then the theorem remains valid in this case as well.

2) From Theorem 6 and equation (16) it follows that for a triangle homeomorphic to the disc, of the three differences \(\tilde{\alpha} - \alpha_0, \quad \tilde{\beta} - \beta_0, \quad \tilde{\gamma} - \gamma_0, \)
not only the sum of the positive differences does not exceed \( \omega^+ \), but also the sum of the absolute values of the negative differences does not exceed \( \omega^- + \tau^- \) where \( \tau^- \) is the negative part of the rotation of the sides (on the side of the triangle).

3) Since we are now dealing with triangles excised from a manifold, evidently the assertion of the lemma is also valid for a “triangle” whose sides are not shortest arcs in the entire polyhedral metric, but only do not admit a shorter joining of the vertices in the “triangle” itself, i.e. which are “relative” shortest arcs in the triangle.

15. Arcs and chords in a polyhedral metric.

Theorem 7. If on the plane two points \( A \) and \( B \) are joined by a segment \( l \) and a polygonal line \( L \), while \( l \) and \( L \) form together a simple closed curve, and for the rotation of \( L \) on any of its segments \( XY \) the condition \( |\tau(XY)| \leq \tau_m < \pi \) is satisfied, then the lengths of \( L \) and \( l \) are connected by the inequality

\[
 l \geq L \cos \frac{\tau_m}{2}. \tag{21}
\]

Proof. We shall consider the links of \( L \) as vectors (Figure 34). We rearrange them by parallel translation in the order of succession of their directions, where as the first among directions on the plane we take the direction \( \overrightarrow{BA} \), and so forth clockwise. The resulting polygonal line \( L' \) (Figure 35) also has the length \( L \), goes from \( A \) to \( B \), but has nonnegative right rotation at all of its vertices. We assert that the rotation of this entire polygonal line, equal to the rotation from its first vector \( r_1 \) to the last vector \( r_2 \), does not exceed \( \tau_m \).

Indeed, consider the vectors \( r_1, r_2 \) in the structure of original polygonal
line $L$. If $r_1$ preceded $r_2$ (Figure 36a), then the rotation from $r_1$ to $r_2$ equals $\tau(X,Y)$. But if $r_2$ was first in the structure of $L$, then the rotation from $r_1$ to $r_2$ is equal to $\tau(X'Y')$ and again does not exceed $\tau_m$.

![Figure 36.](image)

It remains to prove the assertion of the lemma for the polygonal line $L'$. But it is convex and has a rotation not larger than $\tau_m < \pi$. Therefore it does not decrease if we pass to the two-link broken line made up of the extensions to intersection of the end-links of $L$ (Figure 35). This does not change the rotation of the polygonal line. For the two-link polygon thus obtained the lemma is obvious, since among all triangles with a given base $l$ and vertex angle $(\pi - \tau_m)$ the largest sum of the lateral sides is attained for the isosceles triangle, for which this sum is equal to $l \cos^{-1}(\tau_m/2)$.

**Corollary.** If for the rotation $\tau = \tau^+ - \tau^-$ of the polygonal line $L$ on the side of the region enclosed by $L + l$, the quantity $\tau^+ < \pi$, then

$$l \geq L \cos \frac{\tau^+}{2}.$$  

Indeed, in this case $\tau^+ = \tau^- + \alpha + \beta$ (Figure 34). Therefore $\tau^- \leq \tau^+$ and $\tau_m = \sup \tau(XY) \leq \tau^+ < \pi$, and therefore inequality (22) follows from (21).

**Theorem 8.** If in a polyhedral metric two points $A$ and $B$ are joined by a shortest arc $l$ and a broken line $L$, while these curves form a simple closed curve, which bounds an open region $G$ homeomorphic to the disc, then for $\tau^+ + \omega^+ < \pi$

$$l \geq L \cos \frac{\tau^+ + \omega^+}{2},$$

where $l$ and $L$ are the lengths of the curves, $\tau^+$ is the positive part of the rotation of the broken line $L$ on the side of $G$, and $\omega^+$ is the positive part of the rotation of the region $G$.

---

4 In inequality (22) the case of equality holds only for the two-link broken line $L$ forming with $l$ an isosceles triangle. In (21) equality is attained for infinitely many "saw-toothed" polygonal lines.
As above, the region $\overline{G}$ with the boundary included may be considered as convex. We shall join the point $A$ by shortest arcs to the successive vertices $X_1, X_2, \cdots$ of the broken line $L$ (Figure 37). We draw each shortest arc in $\overline{G}$, and we single out the leftmost of the possible shortest arcs. Then $\overline{G}$ is seen to be divided into nonoverlapping reduced triangles $AX_iX_{i+1}$.

On the plane we shall successively construct triangles with the same sides, and adjoin them to one another just as they were adjoined in the structure of $\overline{G}$. If some one of the shortest arcs $l_{i+1}$ coincides with $l_{i+1}X_{i+1}$, then the corresponding triangle degenerates into a segment. In making a circuit we simply drop such triangles. After going around all the triangles, we obtain a closed polygon on the plane, consisting of the segment $AB$ and a polygon $L_0$ of length $L$ (Figure 38). It is not hard to show that the positive part $\tau^+_0$ of the rotation of the plane broken line $L_0$ does not exceed $\tau^+ + \omega^+$.

Indeed, consider any vertex $X_i$ at which the rotation of the broken line $L_0$ is positive: $\alpha_0 + \beta_0 < \pi$ (Figure 38). We have

$$\tau^+_0(X_i) = \pi - (\alpha_0 + \beta_0) = \tau(X_i) + \overline{\alpha} - \alpha_0 + \overline{\beta} - \beta_0,$$

where $\tau(X_i)$ is the rotation of the broken line $L$ in the original polyhedral metric at the point $X_i$.

Adding all these inequalities for the vertices $X_i$ with positive rotations $\tau^+_0(X_i)$, we obtain

$$\tau^+_0(L_0) = \sum \tau(X_i) + \sum (\overline{\alpha} - \alpha_0) + \sum (\overline{\beta} - \beta_0).$$

But the first of the sums on the right does not exceed $\tau(L)$, and the remaining two, from Theorem 6, do not exceed $\omega^+$. Therefore

$$\tau^+_0 \leq \tau^+ + \omega^+.$$
In order to obtain (23), it remains to apply the inequality (22) to $L_0$.

6. **Approximation by polyhedral metrics.**

16. **Approximation in a polygon with bounded excesses.** Consider a polygon $P$, triangulated into triangles $T$ homeomorphic to the disc, in each of which none of the sides is equal to the sum of two others. With respect to the triangles $T$ construct plane triangles $T_0$ with the same lengths of sides, and construct from them a development $P_0$ in which the triangles $T_0$ adjoin one another in the same order as the triangles $T$ in the structure of $P$. The development $P_0$ is a metric space with a polyhedral metric $\rho_0$ determined by the development itself.

The vertices and the points of the edges of the development are naturally associated with the vertices and points on the edges of the triangulation (in accordance with the length along the edges). This correspondence may be extended in an arbitrary way inside the triangles so as to preserve the 1-1 character, and the continuity in both directions, of the mapping of $P_0$ onto $P$. After this the polyhedral metric $\rho_0$ may be considered as given on pairs of points of the manifold $P$ itself.

**Theorem 9.** Suppose that the convex polygon $P$ is triangulated into simple triangles, while the vertices of the triangulation which are not the original vertices of $P$ lie at points through which there pass shortest arcs, and none of the sides of any triangle is equal to the sum of the two others. Then if $C(P) < \pi/7$, the distance $\rho$ between any pair of points $X, Y \in P$ in the polygon $P$ and the distance $\rho_0$ between the same pair of points (more precisely their images) in the development $P_0$ constructed from that triangulation are connected by the inequality

\[
-C_2 d \leq \rho_0 - \rho \leq C_1 d.
\]

Here $d$ is the largest of the diameters of the triangles of the triangulation in the original metric and $C_1$ and $C_2$ are constants depending only on $P$. As $C_1$ and $C_2$ one may employ the quantities

\[
C_1 = 2 + 5C(P) + (n - 2\chi)\pi,
\]

\[
C_2 = 2 + 5C(P) + (n - 2\chi)\pi + \frac{49}{8} C(P)^2,
\]

where $n$ is the number of vertices and $\chi$ the Euler characteristic of the polygon $P$.

Before proceeding to the proof of Theorem 9, we shall establish two auxiliary propositions.
Lemma 18. For the difference between the angle \( \alpha \) in a triangle \( T \) which is convex relative to the boundary and the corresponding angle \( \alpha_0 \) of the plane triangle with the same sides, the estimate

\[
\alpha_0 - \alpha \leq 2C(T) - \delta(T)
\]

holds.

Indeed, if \( \alpha, \xi, \) and \( \eta \) are the angles of the triangle \( ABC \), then

\[
\alpha_0 - \alpha = (\xi - \xi_0) + (\eta - \eta_0) - \delta(T) \leq \nu_1^* + \nu_2^* - \delta(T) \leq 2C(T) - \delta(T),
\]

which proves the lemma.

Now suppose that we are given a triangle \( T = OAB \) which is convex relative to the boundary. Let its sides be \( a,b, \) and \( c, \) and suppose that its diameter is \( d. \) Suppose that points \( X,Y \) are marked off on its sides \( OA \) and \( OB \) respectively, distant from \( O \) by the distances \( x,y. \) Suppose that \( z \) is the distance between these points and \( T' \) a reduced triangle excised from \( T \) by the leftmost shortest arc \( XY \) passing in \( T. \) Consider the plane triangle \( T_0 \) (Figure 39) with sides \( a,b,c. \) On its sides \( OA, OB \) we mark off two points \( X',Y' \) at the distances \( x,y \) from \( O. \) Suppose that \( X'Y' = z_0. \)

Lemma 19. For the distance \( z - z_0 \) the following estimates hold:

\[
(z - z_0) \leq [3C(T) - \delta(T')]d \quad \text{for} \quad z \geq z_0,
\]

\[
(z_0 - z) \leq [3C(T) - \delta(T)]d \quad \text{for} \quad z < z_0.
\]

Proof. We apply the plane triangle \( OXY \) with sides \( x,y,z \) to the segment \( OX' \) of the side \( OA, \) as in Figure 39. If \( z \geq z_0, \) then from the triangle \( XYY' \) and the isosceles triangle \( OYY', \) and with the notations indicated by Figure 39 we have:

\[
(z - z_0) = YY' = 2y \sin \frac{\gamma(x,y) - \gamma(a,b)}{2} \leq d[\gamma(x,y) - \gamma(a,b)],
\]

and for \( z < z_0 \)

\[
(z_0 - z) \leq d[\gamma(a,b) - \gamma(x,y)].
\]

But, taking into account the angle \( \alpha \) at the vertex \( O \) of the triangle \( T \) itself, we have
\[\alpha - \gamma(a, b) \leq \nu_0 \leq C(T),\]

and from Lemma 18
\[\gamma(x, y) - \alpha \leq 2C(T') - \delta(T') \leq 2C(T) - \delta(T').\]

Along with the inequality (30) this yields (28).

Analogously,
\[\gamma(a, b) - \alpha \leq 2C(T) - \delta(T),\]
\[\alpha - \gamma(x, y) \leq 2C(T') \leq 2C(T),\]

from which, taking (31) into account, we obtain the inequality (29).

Lemma 19 is proved.

We turn to the proof of Theorem 9.

1. We join \(X\) and \(Y\) by a shortest arc \(L\) in the metric \(\rho\). Because of the convexity of \(P\) and of all the triangles \(T\), we may assume that \(L\) lies entirely inside \(P\) and moreover that in each triangle \(T\) it has no more than one segment and passes from one side to the other as the leftmost shortest arc in that triangle.

Mark on \(L\) the ends of the segments \(z\), each of which lies in one triangle \(T\). In the development \(P_0\) we mark off points corresponding to these points and join them by segments \(z_0\) lying in the plane triangles \(T_0\). We obtain a curve \(L_0\) joining \(X\) and \(Y\) in \(P_0\), which consists of segments \(z' + z'' + \sum z_0\), where \(z'\) and \(z''\) are its terminal segments from the points \(X, Y\) to the first intersections with the edges of the development.

Evidently we have
\[\rho(XY) = z' + z'' + \sum z,\]

where \(z'\) and \(z''\) are the terminal segments of the curve \(L\) from \(X\) and \(Y\) up to the first intersections of \(L\) with the edges of the triangulation. Further we have
\[\rho_0 - \rho \leq [z_0 - z', z'' - z''] + \sum (z_0 - z) \leq 2d + \sum (z_0 - z).\]

The inequality is merely strengthened if we preserve only the positive terms in the last sum. We shall denote the sum over these terms by \(\sum'\). For each of these terms one may use the estimate (29), which yields
\[\rho_0 - \rho \leq d[2 + 3 \sum' C(T) - \sum' \delta(T)].\]

Finally, in \(\sum' \delta(T)\) we drop the positive terms, and for the sum of the remaining terms we use estimate \(8'\) from subsection 11. This leads to the right inequality in (24), Theorem 9:
\[\rho_0 - \rho \leq [2 + 5C(P) + (n + 2\chi) \pi]d.\]
In obtaining this inequality the restriction \( C(P) < \frac{\pi}{7} \) played no role.

2. We turn to the proof of the left of the inequalities \((24)\). We join \( X \) to \( Y \) by a shortest arc \( L_0 \) in the development \( P_0 \). Since in the structure of \( P_0 \) the triangles \( T_0 \) are not necessarily convex, we may not assume this time that \( L_0 \) enters each of the \( T_0 \) no more than once. This does not allow us to estimate the difference \( \rho - \rho_0 \) in the same order as we just did for \( \rho_0 - \rho \).

Let us follow the course of the shortest arc \( L_0 \) on the path from \( X \) to \( Y \). Suppose that \( L_0 \) enters some triangle \( T \) through the side \( a \). If \( L_0 \) intersects \( a \) multiply, we obtain a series of two-gons \( D \) formed by the segments of \( L_0 \) and pieces of the side \( a \). We assert that no pair of two-gons of this kind can, in view of the inequality \( C(P) < \frac{\pi}{7} \), overlap one another.

Indeed, the shortest arc \( L_0 \) is itself a simple arc. Therefore for the two-gons to intersect it is necessary that the curve \( L_0 \) must run into the two-gon \( AB \) a second time, intersecting the arc \( AB \), which is a segment of the side \( a \), from outside (Figure 40). Then, however we obtain a new two-gon \( AC \). Suppose that \( \omega \) is the curvature of its interior region, \( \bar{\alpha} \) and \( \bar{\lambda} + \bar{\mu} \) the angles (indicated on Figure 40) of the sectors at its vertices, \( \tau_0 = -\tau_\kappa \) the rotation of the portion \( AC \) of the shortest arc \( L_0 \) from the side of the two-gon \( AC \), \( \tau, \tau^+, \tau^- \) the characteristics relating to the rotation of the side \( a \).

For the two-gon \( AC \) the Gauss-Bonnet theorem gives, in the polyhedral metric,

\[
(\pi - \bar{\alpha}) + \tau_\kappa + (\pi - \bar{\lambda} - \bar{\mu}) + \tau + \omega = 2\pi,
\]

so that

\[
\omega \geq \bar{\alpha} + \bar{\lambda} + \bar{\mu} + \tau_\kappa - \tau^+(AC) + \tau^-(AC)
\]

\[
\geq \bar{\lambda} - \tau^+(AC) \geq \pi - \tau^+(C) - \tau^+(AC),
\]

or in other words

\[
(32) \quad \omega + \tau^+(C + AC) \geq \pi.
\]

But by Lemma 14
\[ \omega \leq \omega^+(P_0) \leq 4C(P), \]

and by Lemma 15
\[ \tau^+(C + AC) \leq \tau^+(a) \leq 3C(P). \]

Therefore the relation (32) is necessary when \( C(P) < \pi/7 \).

Thus the two-gons formed by \( L_0 \) with the sides from \( a \) have no common points.

3. Consider one such two-gon \( D \). Suppose that \( \omega^+ \) is the positive part of the curvature of the interior region of \( D \), and \( \tau^+ \) is the positive part of its side \( AB \), a portion of \( a \). Then we have \( \omega^+ + \tau^+ \leq 4C(P) + 3C(P) < \pi \). This makes it possible to apply Theorem 8 to each two-gon \( D \), as a consequence of which the length \( A_0B_0 \) of the side of \( D \) which is a segment of the shortest arc \( L_0 \) and the length \( AB \) of its other side are connected by the relationship

\[ A_0B_0 \geq AB \cos \frac{\omega^+ + \tau^+}{2}. \]

Hence
\[ AB \leq AB \left(1 - \cos \frac{\omega^+ + \tau^+}{2}\right) + A_0B_0 \]

and
\[ AB - A_0B_0 \leq AB \cdot 2 \sin^2 \frac{\omega^+ + \tau^+}{4} \leq \frac{d}{8} (\omega^+ + \tau^+)^2 \]
\[ \leq \frac{d}{8} \left[ \omega^+(P_0) + \sum \tau^+ \right] (\omega^+ + \tau^+), \]

so that finally
\[ AB - A_0B_0 \leq \frac{7d}{8} C(P) (\omega^+ + \tau^+). \]

4. We shall follow along the shortest arc \( L_0 \). In the case when it intersects the side of the triangle \( T_0 \) several times, we replace it on the portion of each of the two-gons thus formed by the corresponding portion of the side of \( T_0 \), and then we proceed along \( L_0 \). When we have completed this substitution, we have obtained a new curve \( L'_0 \) which is longer than \( L_0 \) by not more than

\[ L'_0 - L_0 \leq \frac{7d}{8} C(P) \sum D (\omega^+ + \tau^+) \leq \frac{49d}{8} C(P)^2. \]

The curve \( L'_0 \) consists of pieces going along the sides of the triangles \( T_0 \) and pieces going inside these triangles. We cannot have more than
two pieces of the latter type in each of the $T_0$, since $L_0'$ would then repeatedly intersect one of the sides of the triangle.

Now we mark off in $P$ the ends of the pieces of $L_0'$ on the curves of the triangulation, and join them by shortest arcs in $P$. We obtain a curve $L$. We have

$$\rho - \rho_0 \leq L - L_0 = L - L_0 + L_0' - L_0$$

$$\leq \left[ 2d - \sum (z - z_0) \right] + \frac{49d}{8} C(P)^2,$$

where $z$ and $z_0$ are lengths of the corresponding pieces of the structure of $L$ and $L_0'$ from which $z_0$ goes inside $T_0$.

Keeping only the positive differences $(z - z_0)$ and taking inequality (28) into account, we have:

$$\sum (z - z_0) \leq \left[ 3C(P) - \sum \delta^{-}(T') \right] d,$$

where $T'$ are the triangles excised from $T$ by the shortest arcs $z$.

Taking into account the fact that the triangles $T'$ form only a portion of some other triangulation $P$, in which, from the remark to Lemma 13, we have the right to apply the result $(8')$,

$$\sum \delta^{-}(T') \leq 2C(P) + (n - 2\chi) \pi,$$

we finally obtain

$$\rho - \rho_0 \leq \left[ 2 + 5C(P) + (n - 2\chi) \pi + \frac{49}{8} C(P)^2 \right] d.$$

This proves the left inequality in (24) and completes the proof of Theorem 9.

17. Approximation of the metric in an arbitrary region.

Theorem 10. In a two-dimensional manifold with metric $\rho$ and of bounded curvature within the limits of any convex polygon $P$, the sectors at the vertices of which are convex relative to the boundary or consist of pieces which are convex relative to the boundary, the metric $\rho$ is the limit of a uniformly convergent sequence of polyhedral metrics $\rho_0$ with respect to which the absolute curvatures are uniformly bounded.

This theorem differs from the preceding one by the absence of the restriction $C(P) \leq \pi/7$.

Proof. 1. Once again we shall need a further specialization of the choice of the triangulations with respect to which the polyhedral metrics $\rho_0$ converging to $\rho$ will be constructed.
Consider some triangulation of the polygon $P$ into simple triangles $T$ of diameter less than $\varepsilon_1$. We suppose that in these triangles no side is equal to the sum of the other two and that the vertices other than the vertices of $P$ lie at points through which there pass shortest arcs. Among these triangles there are no more than $N_1 = (14/\pi)C(P)$ with $C(T) \geq \pi/14$. These triangles will be called singular.

2. Suppose further that $N_2$ is the total number of vertices in the triangulation just chosen. In the neighborhood of each of the $N_2$ vertices we select a portion of the curve of the triangulation of length $\varepsilon_2$ and adjacent to that vertex (Figure 41). The remaining segments $a$ of the edge form closed sets with no common points. The smallest of the distances between these sets in the metric $\rho$ will be denoted by $\delta$, which is obviously positive.

3. Each of the nonsingular triangles $T$ will be subdivided into very fine simple triangles $t$ of diameter less than $\varepsilon_3$. We suppose that in each of them no side is equal to the sum of the other two and that the vertices other than the vertices of $T$ lie at points through which there pass shortest arcs. Then we consider a development $P_0$ formed of plane triangles $t_0$ and $T_0$ corresponding to the triangles $t$ and the singular triangles of $T$. To the nonsingular triangles $T$ in this development there correspond certain polyhedral regions $T_0$, which for simplicity we shall continue to call triangles.

4. We assert that for any $\varepsilon > 0$ the quantities $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ may be successively chosen that for the metric $\rho_0$ of the development $P_0$ and the original metric $\rho$ the inequality

$$\rho - \rho_0 < \varepsilon$$

is satisfied for any pair of points $X, Y \in P$.

Since the opposite estimate $\rho_0 - \rho < \varepsilon$ follows from the right inequality in (24), which is valid also under the hypotheses of Theorem 10, then the entire proof of the theorem reduces to obtaining the estimate (33). We now turn to that proof.

5. Suppose that $X$ and $Y$ are any two points of $P$. We join the corresponding points $X$ and $Y$ in $P_0$ by a shortest arc $L_0$ in the polyhedral metric $\rho_0$. If $\rho_0 \geq \rho$, then (33) is satisfied. Hence we may suppose $\rho_0 < \rho \leq D$,
where $D$ is the diameter of the polyhedron $P$ in the metric $\rho$. (33) is satisfied also if the points $X$ and $Y$ lie in a single triangle $T$ (for $\varepsilon_1 < \varepsilon$). Therefore we may suppose further that $X$ and $Y$ belong to distinct triangles $T$.

6. The shortest arc $L_0$ may be divided into segments of six types, where we assume, when singling out pieces of each successive type or the next piece of a given type that all the pieces already singled out have been so to speak removed from the structure of $L_0$. We shall make no special mention of this in what follows. For each distinguished piece we mark off in the polygon $P$ points corresponding to the ends of this piece, and join them by shortest arcs in the metric $\rho$. We denote by $L_0^i$, $i = 1, \cdots, 6$, the combined length of the pieces of type $i$ of the curve $L_0$ in the development $P_0$, and by $L'$ the combined length of the shortest arcs in $P$ constructed from these pieces.

7. To type 1 we assign the endpieces from $X$ and $Y$ up to the first intersections with the sides of the triangles $T_0$. There are always two of these pieces, so that

$$L^1 - L_0^1 \leq L^1 \leq 2\varepsilon_1 < \frac{\varepsilon}{6}.$$ 

The last of the above inequalities is taken to be satisfied by the choice of $\varepsilon_1$.

8. Suppose that as we follow along $L_0$ from $X$ to $Y$ the shortest arc $L_0$ first encounters one of the singular triangles at the point $A$. Then we select the piece from $A$ to the last common point of $L_0$ and $T_0$ and then proceed along $L_0$. We assign pieces obtained in this way to type 2. They are less than $N_1$ in number, so that

$$L^2 - L_0^2 \leq L^2 \leq N_1 \varepsilon_1 < \frac{\varepsilon}{6}.$$ 

The last of these inequalities may be ensured by the choice of $\varepsilon_1$.

9. Near each vertex of the original subdivision of $P$ into triangles $T$ there have been singled out pieces of the edges of the triangulation of length $\varepsilon_2$ each. These pieces form a bundle of segments converging at a given vertex. If the curve first closed on such a bundle with vertex $O$ at the point $B$, then we single out the portion from $B$ to the last point $C$ of the intersection of $L_0$ with that bundle, and then follow further along $L_0$. The pieces singled out in this way are assigned to type 3. They are not greater than $N_2(N_1 + 1)$ in number, so that
\[ L^3 - L_0^3 \leq L_0^3 \leq N_2(N_1 + 1)2\varepsilon_2 < \frac{\varepsilon}{6}. \]

The last of these inequalities may be ensured by the choice of \( \varepsilon_2 \).

10. After we have removed the pieces of types 1, 2, and 3, the remaining portion of \( L_0 \) decomposes into not more than \( (N_1 + 1)(N_2 + 1) \) pieces. Each of these pieces has no interior points in common with the singular triangles \( T_0 \) or with the bundles at the vertices of the triangulation, i.e., the sides of all the triangles \( T_0 \) can only be intersected in segments \( a \). We assign to type 4 the end-portions of the remaining pieces running from the ends of these pieces to the first intersections with the segments \( a \). Evidently these pieces do not number more than \( 2(N_1 + 1)(N_2 + 1) \) elements, and each of them runs in a single nonsingular triangle \( T_0 \). Therefore

\[ L^4 - L_0^4 = \sum (z - z_0) \leq 2(N_1 + 1)(N_2 + 1) \max (z - z_0), \]

where \( z_0 \) is the length of one piece of this type and \( z \) the length of the corresponding shortest arc in the metric \( \rho \).

Inasmuch as each piece of type 4 runs in a nonsingular triangle \( T_0 \), Theorem 9 is valid for the triangle \( T \) corresponding to it, and we may suppose that the subdivision of \( T \) into triangles \( t \) is as fine as desired, i.e. \( \varepsilon_3 \) is so small that

\[ z - z_0 < \frac{1}{2(N_1 + 1)(N_2 + 1)} \frac{\varepsilon}{6}. \]

Then

\[ L^4 - L_0^4 < \frac{\varepsilon}{6}. \]

11. We assign to type 5 those pieces of the remaining portions of the shortest arc \( L_0 \) on which it goes from the intersection with one segment \( a \) to an intersection with another segment \( a \). Each such segment has a length not less than \( \delta/2 \), and inasmuch as also the entire length \( L_0 < D \), the number of such pieces does not exceed \( 2D/\delta \). Hence

\[ L^5 - L_0^5 = \sum (z - z_0) \leq \frac{2D}{\delta} \max (z - z_0) < \frac{\varepsilon}{6}. \]

Here again \( z_0 \) is the length of one piece and \( z \) the length of the corresponding shortest arc in \( P \). The last inequality, as before, may be ensured from Theorem 9 by proper choice of small \( \varepsilon_3 \).

12. We assign the remaining pieces to type 6. Their number is no larger than \( (N_1 + 1)(N_2 + 1) + D/\delta \). On each of them the shortest arc \( L_0 \) has a
common point only with one segment $a$. On the remainder it does not leave the triangles $T_6^i$, $T_6^j$ adjacent along $a$.

Comparing the length $z_0$ of one piece of type $6$ with the length of the segment $a$ joining its endpoints, i.e. of the corresponding piece $z$, we may use the same argument as in the proof of Theorem 9, since $C(T') + C(T') < \pi/7$. This allows us to take $\varepsilon_3$ so small that

$$L_6^6 - L_0^6 \leq \left[ (N_1 + 1)(N_2 + 1) + \frac{D}{6} \right] \max (z - z_0) < \frac{\varepsilon_3}{6}.$$ 

13. Now the validity of inequality (33), and along with it Theorem 10, is established quite easily:

$$\rho - \rho_0 \leq L - L_0 = \sum_{i=1}^{6} (L^i - L_0^i) < \varepsilon.$$ 

From Theorem 10 it is not difficult to arrive at the following assertion.

**Theorem 11.** On every polygon $P$ consisting of polygons which are convex relative to the boundary, the metric $(\rho)_P$, induced by the distinguishing of $P$ in the whole manifold, is the limit of uniformly converging polyhedral metrics with respect to which the absolute curvatures are uniformly bounded.

**Remarks.** 1) Theorem 10 is a special case of Theorem 11, since in the case of convexity the metrics $\rho_P$ and $\rho$ coincide.

2) From the existence at each point of a neighborhood in the form of an absolutely convex polygon (Theorem 1) and from Theorem 10 follows the possibility of local approximation of the metric $\rho$ by polyhedral metrics for which the absolute curvature is bounded uniformly (this last in view of Theorem 5).

3) Since any compact set can be covered by a polygon of the type indicated in Theorem 11, we arrive at the following result: every compact set $M$ can be imbedded in a polygon $P$ such that the metric $(\rho)_P$ is the limit of a uniformly convergent sequence of polyhedral metrics for which the absolute curvature is bounded uniformly.

4) In the proof of Theorem 10 we did not keep to full analogy with the proof of the similar assertion in [5]. The proof presented there on page 252 is not completely exact. Indeed, using Lemma 3 proved in [5], one may assert only that, within the limits of a triangle $T$ with $\omega^+(T) < \pi$, it is not the metrics $\rho_i$ themselves which converge to the metric $\rho$, but rather the metrics $(\rho_i)_T$ induced by distinguishing from the development that piece of it which corresponds to $T$. The proof of Theorem 10 presented
above includes the case of a metric of positive curvature and thus eliminates the indicated inexactness.

In Theorems 9 and 10 the uniform smallness of the difference $p - p_0$ for small diameters $d$ of the triangles of the triangulation was guaranteed by the given specialization of the triangulation. It would be interesting to answer the following questions. Would not the difference $|p - p_0|$ be small along with $d$ for any triangulation with respect to which it is possible to construct a development? Which exact constants replace the rough constants $C_1$ and $C_2$ which we have chosen? What better values of $C_1$ and $C_2$ might we achieve by specializing the triangulation and the mappings of $P$ onto $P_0$ inside the triangles of the subdivision?
Chapter IV

Metrics Admitting Approximation by Polyhedral Metrics

1. Convergent metrics and the limit metric.

1. Object of the investigation. The object of the present chapter is the investigation of two-dimensional manifolds which have intrinsic metrics admitting, at least locally, uniform approximation by polyhedral metrics for which the positive part of the curvature is uniformly bounded. In other words we suppose that the metric space under investigation is a two-dimensional manifold, has an intrinsic metric, and for each point $O$ there is a neighborhood $G$, homeomorphic to the disc, of that point, and also a sequence of manifolds $P_n$ with polyhedral metrics $\rho_n$ and homeomorphic mappings $\phi_n$ of the regions $G_n \subset P_n$ onto the region $G$. These mappings carry the metrics $\rho_n$ onto the region $G$. We suppose that the positive part of the curvature of the polyhedral metric $\rho_n$ in the region $G$ is bounded uniformly, and that the metrics $\rho_n$ converge uniformly to $\rho$ in the region $G$.

In the preceding chapter we have proved that two-dimensional manifolds with an intrinsic metric certainly admit such approximation. Moreover, it was proved that in a two-dimensional manifold with an intrinsic metric $\rho$ of bounded curvature each point has a small neighborhood in the form of a convex polygon homeomorphic to the disc. This polygon can play the role of the region $G$ indicated above, and the polyhedral metrics $\rho_n$ approximating $\rho$ may be chosen to have bounded absolute curvature and not just bounded positive curvature.

In this chapter we prove the converse assertion: each two-dimensional manifold with an intrinsic metric locally admitting uniform approximation by polyhedral metrics for which the positive part of the curvature is bounded uniformly has a metric of bounded curvature.

This makes it possible in what follows, depending on the needs of the situation, to begin either from the intrinsic properties of the metric itself, or from a consideration of the approximating polyhedral metrics. Thus, beginning from the approximation of the metric by polyhedral metrics, we establish the existence of an angle between arbitrary shortest arcs,
the boundedness of the angles of sectors and so forth. Moreover, we obtain a strengthening of the condition of boundedness of the curvature: it is proved that in a region $G$ with a compact closure, for a finite system of arbitrary nonintersecting reduced triangles, and not just simple triangles, the sum of the absolute values of their excesses is a bounded number depending only on the choice of $G$. On the other hand, for a space which is locally approximable by some sort of polyhedra, the fact that we have established the boundedness of the curvature of this space gives us the right to consider the particularly convenient approximations by polyhedra constructed in Chapter III with respect to the metric of the space itself.

The object of the next two subsections is to explain the concept of convergence of metrics.

2. Possible peculiarities. 1. A function $\rho$ which is a limit of the metrics $\rho_n$ is not in general necessarily a metric. (The condition that $\rho(X,Y) = 0$ only when $X = Y$ may fail to hold.)

2. The metrics $\rho_n$ may converge nonuniformly to a limit metric.

Example. Suppose that $R$ is the unit plane disc with center at the origin of coordinates, $\rho$ is its Euclidean metric, and $\rho_n$ is the intrinsic metric of the surface obtained from $R$ by removing a disc of radius $1/2n$ and center at the point $(1/n, 0)$ and replacement of the hole by a cone of height 1. We transfer $\rho_n$ to $R$ by simple projection. Evidently $\rho_n \Rightarrow \rho$, but the convergence is not uniform.

3. The topological space into which the set $R$ is made by the metric $\rho$ will be denoted in what follows by $R(\rho)$. We shall say that the metrics $\rho_n$ define in $R$ one and the same topology if they define in $R$ one and the same system of open sets. In other words, $R(\rho_n)$ are not only homeomorphic but are identically mapped onto one another already by their homeomorphisms. The limiting metric $\rho$ may, generally speaking, define a topology in $R$ which is different from that defined in $R$ by the metrics $\rho_n$.

Example. Suppose that $R$ is the set of points of the open arc $AB$. As a metric $\rho_n$ we take the distance measured in the Euclidean space containing the curve $AB$. As the endpoint $B$ of the curve approaches to an interior
point $C$ (Figure 42) the metrics $\rho_n$ will uniformly converge to the metric $\rho$, converting $R$ into a different topological space. In fact, in the limiting position, for the points $X_i \subset R$, approaching the end point $B$, we have $X_i \rightarrow C$, while there is no such convergence in the metrics through which the limit is being taken.

4. We call a metric for a space *complete* if each sequence that is convergent to itself in that metric has a limit point. The limit of complete metrics may turn out, generally speaking, to be a noncomplete metric.

We give two examples.

**Example 1.** Suppose that $R$ is the Euclidean plane. To each point $X \in R$ lying at a distance $r$ from the origin of coordinates $O$ we assign a point $X'$ lying on the ray $OX$ and at a distance from $O$ equal to

$$ r' = \begin{cases} \frac{2}{\pi} \arctan r & \text{for } r \leq n, \\ r \frac{2 \arctan n}{\pi n} & \text{for } r > n. \end{cases} $$

We define the metrics $\rho_n(X,Y)$ as the usual Euclidean distance between $X'$ and $Y'$. Evidently the metrics $\rho_n$ are complete. At the same time they converge to a metric, which in the large on the plane $R$ coincides with the incomplete metric of the open Euclidean disc on the usual plane.

**Example 2.** Suppose that $R$ is the segment $(0,1]$. We mark off the points $1/2$, $1/2^2$, $1/2^3$, $\ldots$, and we map this segment onto the segment $[0,1]$ according to the rule

$$ x' = \begin{cases} x & \text{for } x = \frac{1}{2^k} (k = n, n + 1, \ldots), \\ 0 & \text{for } x = \frac{1}{2^n}, \\ \frac{1}{2^{k-1}} & \text{for } x = \frac{1}{2^k} (k = n + 1, n + 2, \ldots). \end{cases} $$

We define the metric $\rho_n(x_1, x_2)$ on $(0,1]$ as the usual distance between $x'_1$ and $x'_2$ on the segment $[0,1]$. Each of the metrics $\rho_n$ coincides with the usual metric of the closed segment $[0,1]$ and is complete. At the same time the metrics $\rho_n$ converge uniformly to the incomplete metric of the segment $(0,1]$.

In the first example the loss of completeness was connected with the nonuniform convergence of the metrics. In the second it was connected with the circumstance that the identical mappings of the spaces $R(\rho_n)$
onto one another were not homeomorphisms, although the metrics $\rho_n$ converted $R$ into spaces homeomorphic to the disc.

5. A space is called *boundedly compact* if each set bounded in the sense of the metric of that space has a compact closure.

Generally speaking, even in a metrized manifold, the completeness of the metric does not imply bounded compactness. Suppose for example that $\rho(X,Y)$ is the metric of the Euclidean plane. Consider the metric $\rho^*(X,Y) = \min[\rho(X,Y), 1]$. In the metric $\rho^*$ the plane is complete but is not a boundedly compact manifold. This peculiarity is connected with the fact that the metric $\rho^*$ is not intrinsic.

**Lemma 1.** In a locally compact space with an intrinsic metric, completeness is a necessary and sufficient condition for bounded compactness.

**Proof.** The completeness of the space always follows from the bounded compactness. Suppose now that the space is complete. Consider points $X_n (n = 1, 2, \cdots)$, which lie at uniformly bounded distances $r_n$ from the point $O$. Keeping to the successive order of these points, we may suppose that $r_n \to r$. Because of the intrinsic character of the metric there exist curves $L_n = OX_n$ whose lengths $L_n \to r$. The initial portion of $L_n$ adjacent to $O$ lies in a compact neighborhood $V$ of the point $O$. On these pieces the $L_n$ may be assumed to be shortest arcs. Within the limits of $V$ the pieces $L_n$ converge to some curve $L$. Suppose that $X_0$ is its endpoint. The point $X_0$ in its turn has a compact neighborhood, in which also, by changing the curves somewhat and selecting a subsequence of numbers, we may regard the pieces of $L_n$ as shortest arcs converging to an extension of the curve $L$. Moreover, $X_0$ has moved off from $O$. After a countable repetition of such shifts the points $X_0$ form a sequence convergent in itself, which converges to a new point $X_0$, with respect to which we may continue the shift. Thus we extend the convergence to the entire extent of the curves $L_n$ and establish that a subsequence of the points $X_n$ converges to some point $X$.

Lemma 1 makes it possible, in the two-dimensional manifolds of interest to us, to make no distinction between the properties of completeness and bounded compactness.

6. The limit of intrinsic metrics may turn out not to be an intrinsic metric.

**Example.** Let $R$ be the plane square $0 < x < 1$, $0 \leq y \leq 1$. We introduce a surface $F_n$ as follows. On the part of $R$ where $x \geq 1/n$, we construct
a “roof” with a total length of sloping sides equal to 3, and over the remaining portion of the square we complete it by a slanted side, as depicted in Figure 43. We define the metric $\rho_n$ as the distance on the surface $F_n$ between the points lying over $X$ and $Y$. As $n \to \infty$ the slope of the lateral side of the roof increases. The limiting metric $\rho$ of the metrics $\rho_n$ may be represented as the result of measuring the distances on the surface $F$ with a vertical side instead of a slanted side on the roof. We note that the convergence of the metric $\rho_n \to \rho$ is uniform.

It is easy to see that the limit metric $\rho$ on the set $R$ is not intrinsic. The points $X(1/4, 0), Y(1/4, 1)$, for example, cannot be joined, without leaving $R$, in the metric $\rho$ by a curve close in length to

$$\rho(X, Y) = \lim_{n \to \infty} \rho_n XY \leq 1.5.$$ 

The loss of the intrinsic character of the metric is connected in this case with the incompleteness of the metrics $\rho_n$.

3. The properties of the converging metrics and limit metrics. Evidently the limit $\rho$ of the metrics $\rho_n$ will be a metric if and only if the equality $\rho(X, Y) = 0$ holds only when $X = Y$. Indeed, the remaining conditions imposed on the metric are automatically preserved by the limit metric.

**Lemma 2.** For the uniform convergence of the metrics $\rho_n \to \rho$ it is necessary, and if $R(\rho)$ is compact it is also sufficient, that $\rho_n(X_n, Y_n)$ should converge to $\rho(X, Y)$ for any $X_n \to X, Y_n \to Y$ in $R(\rho)$.

The necessity is proved by the following inequalities, valid for any $\varepsilon < 0$ and for sufficiently large $n$:

$$|\rho_n(X_n, Y_n) - \rho(X, Y)| \leq |\rho_n(X_n, Y_n) - \rho(X_n, Y_n)| + |\rho(X_n, Y_n) - \rho(X, Y)| \leq \varepsilon + \rho(X_n, X) + \rho(Y_n, Y) \leq 3\varepsilon.$$ 

In order to establish sufficiency we suppose that the convergence $\rho_n \to \rho$
is not convergent and that there exist $\epsilon > 0$, $n_i \to \infty$, $X_i, Y_i \in R$ such that

$$|\rho_n(X_i, Y_i) - \rho(X_i, Y_i)| \geq s.$$

Since $R(\rho)$ is compact we may suppose that $X_i \to X$ and $Y_i \to Y$ in $R(\rho)$. Then for sufficiently large $n$ we shall by hypothesis have

$$|\rho_n(X_i, Y_i) - \rho(X, Y)| \leq \frac{\epsilon}{2},$$

so that along with the preceding inequality

$$|\rho(X_i, Y_i) - \rho(X, Y)| \geq \frac{\epsilon}{2},$$

which contradicts the convergence of $X_i$ to $X$ and $Y_i$ to $Y$ in $R(\rho)$.

**Lemma 3.** If the metrics $\rho_n$ define in $R$ one and the same topology and if the convergence $\rho_n \to \rho$ is uniform, and the spaces $R(\rho_n)$ are boundedly compact, then the limit metric $\rho$ defines the same topology and the space $R(\rho)$ is also boundedly compact.

First we shall prove the bounded compactness of $R(\rho)$. Suppose that $M \subset R$ is a set of points bounded in the metric $\rho$. We need to show that one may select a sequence of points from $M$ which converge to a point of $R$. From the uniform convergence $\rho_n \to \rho$ it follows that $M$ is bounded also in the metric $\rho_n$ for sufficiently large $n_0$. In view of the bounded compactness of $R(\rho_n)$ there exists a sequence $X_i \in M$ converging to $X \in R$ in the metric $\rho_n$. In view of the identity of the topologies defined by the metrics $\rho_n$ this sequence converges to $X$ for all $\rho_n$, and therefore, because of the uniform convergence, also in the metric $\rho$.

We need to show that the limit metric $\rho$ defines the same topology in $R$ as the metrics $\rho_n$. For this it suffices to verify that the convergence of $X_i$ to $X$ in any of the metrics $\rho_n$ implies the convergence of $X_i$ to $X$ in the metric $\rho$ and conversely. We have already verified the first of these assertions at the end of the preceding paragraph. It remains to verify the second assertion.

Suppose that $X_i \to X$ in $R(\rho)$. Then all the $X_i$ lie in a region $U \subset R$ which is bounded in the metric $\rho$, and, from the uniform convergence of $\rho_n$ to $\rho$, also bounded in the metric $\rho_n$ for large $n_0$. Because of the bounded compactness of $\rho_n$ there is a subsequence $X_{i_k}$ converging to some point $X_0$ in $R(\rho_n)$ and therefore in all the $R(\rho_n)$. But then, by the uniform convergence of the metric, $X_{i_k} \to X_0$ in $R(\rho)$, from which we conclude that $\rho(X, X_0) = 0$. But since we have assumed that $\rho$ is a metric, we have
$X = X_0$. Hence $X_i \to X$ in $R(\rho_n)$ and in all the $R(\rho)$.

Lemma 3 is completely proved.

**Lemma 4.** Suppose that on the set $R$ the metrics $\rho_n$, defining one and the same topology, converge uniformly to the metric $\rho$. Suppose further that $U$ is a compact part of $R(\rho)$ and that $U$ contains an infinite sequence of curves $L_n$, with the lengths $s_n$ of the curves $L_n$ bounded uniformly in the corresponding metrics $\rho_n$. Then from the curves $L_n$ we may choose a subsequence converging in the metric $\rho$ to some curve $L$. The curve $L$ will be rectifiable in the metric $\rho$ and its length will satisfy $s \leq \lim\inf_{n \to \infty} s_n$.

**Proof.** Because of the fact that they have common topologies, the $L_n$ are curves in any of the metrics. On each of the $L_n$ we select as the parameter $t$ ($0 \leq t \leq 1$) the relative length of the arc: $t = s'/s_n$, where $s'$ is the length of the arc measured from the beginning of the curve. The length for each curve $L_n$ is measured in its metric $\rho_n$. In the interval $[0,1]$ we choose a countable everywhere dense set of values $t_{i}$. Using the compactness of $U$ and a usual diagonal process, we choose a subsequence of the $L_n$ for which $X_i^* \to X_i$ in the metric $\rho$. Here $X_i^* = X^*(t_i)$ are points on the selected curves $L_n$ and the $X_i$ are certain limit points. Since from now on we shall use only the subsequence, we shall keep to the notation $L_n$.

Take an $\varepsilon > 0$. For each $t'$ we select a $t_i$ such that

$$|t' - t_i| < \frac{\varepsilon}{5S},$$

where $S = \sup s_n$. Then for all $n$

$$\rho_n(X^*(t'), X^*(t_i)) < \frac{\varepsilon}{5}.$$

We choose further $N$ so large that for $n,m > N$ the following inequality will hold for that value of $t_i$ and for any pair $X,Y$:

$$\rho(X^*(t_i), X^*(t_{i})) < \frac{\varepsilon}{5}, \quad |\rho(X, Y) - \rho_n(X, Y)| < \frac{\varepsilon}{5}.$$  

Then we have:

$$\rho(X^*(t'), X^*(t')) \leq \rho(X^*(t'), X^*(t_i)) + \rho(X^*(t_i), X^*(t_{i})) + \rho(X^*(t_i), X^*(t'))$$

$$< \rho_n(X^*(t'), X^*(t_i)) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \rho_n(X^*(t_i), X^*(t')) + \frac{\varepsilon}{5} < \varepsilon.$$
Thus the points $X^n(t')$ converge to some limit $X(t')$ for each $t'$. The limit points $X(t)$ form a continuous curve $L$, since

$$\rho_n(X^n(t_1), X^n(t_2)) \leq S|t_1 - t_2|,$$

so that by Lemma 2

$$\rho(X(t_1), X(t_2)) \leq S|t_1 - t_2|.$$

From the last two inequalities it is easy to show that the points of the curves $L_n$ converge to the corresponding points of the curve $L$ in the metric $\rho$ uniformly for all $t$, i.e., $L_n \to L$ in the metric $\rho$. From the same inequality it follows that the limit curve $L$ is rectifiable in the metric $\rho$. Also for any system $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{r+1} \leq 1$

$$s_n \geq \sum_{i=1}^{r} \rho_n(X_i^n, X_{i+1}^n) \to \sum_{i=1}^{r} \rho(X_i, X_{i+1}),$$

from which

$$\liminf_{n \to \infty} s_n \geq \sum_{i=1}^{r} \rho(X_i, X_{i+1})$$

and

$$\liminf_{n \to \infty} s_n \geq s.$$

Lemma 4 is proved.

**Lemma 5.** If the interior metrics $\rho_n$ determine one and the same topology and converge uniformly to the metric $\rho$, and the shortest arcs $L_n$ in $\rho_n$ converge in the metric $\rho$ to the curve $L$, then $L$ is a shortest arc in the metric $\rho$ and its length is equal to the distance $\rho(X,Y)$ between its endpoints.

**Proof.** The equation $s = \rho(X,Y)$ follows from the inequalities

$$\rho(X, Y) \leq s \leq \liminf_{n \to \infty} s_n = \lim_{n \to \infty} \rho_n(X_n, Y_n) = \rho(X, Y).$$

The fact that $L$ is a shortest arc follows from $s = \rho(X,Y)$.

**Lemma 6.** If the metrics $\rho_n$ are intrinsic, the convergence $\rho_n \to \rho$ uniform and all the metrics $\rho_n$, $\rho$ determine one and the same topology, in terms of which the space $R(\rho)$ is locally compact, then the limit metric $\rho$ in the small is also intrinsic.

**Proof.** Choose a point $O \in R$ and around it a compact neighborhood $V$ of radius larger than $2r$ in the metric $\rho$. Moreover, take a neighborhood $U$ of radius $r$. For a sufficiently large $n$ every two points $X,Y \in U$ may be joined in the metric $\rho_n$ by shortest arcs $L_n$ of length $\rho_n(X,Y)$, and also $L_n \subset V$. By Lemma 4, it is possible to select from the curves $L_n$ a sequence
converging in the metric $\rho$ to some curve $L$. By Lemma 5, this length will be equal to $\rho(X,Y)$ and $L$ will be a shortest arc in the metric $\rho$. Thus every two points $X,Y$ of a sufficiently small neighborhood of an arbitrarily chosen point $O$ may be joined in the metric $\rho$ by a shortest curve of length $\rho(X,Y)$, so that the limit metric is intrinsic in the small.

**Lemma 7.** If the intrinsic metrics $\rho_n$, defining in $R$ one and the same topology, uniformly converge to a metric $\rho$ and the spaces $R(\rho_n)$ are boundedly compact, then $\rho$ is also an intrinsic metric and any two points in it can be joined by a shortest arc.

The proof of this lemma is analogous to the proof of Lemma 6 and will not be carried out.

**Remark.** Under the conditions of Lemmas 6 and 7 any two points in a sufficiently small neighborhood may be joined in the limit metric by a shortest arc which is a limit of some subsequence of shortest arcs in the converging metrics $\rho_n$. But we have no right to assert that every shortest arc in the limit metric $\rho$ can be considered as such a limit of shortest arcs from the metrics $\rho_n$. This compels us sometimes to resort to special constructions, making it possible to manage with shortest arcs that are easier to investigate and admit approximations of the indicated sort.

By joining one or another system of points by shortest arcs, we can always, as already mentioned in subsection 2 of Chapter III, avoid superfluous intersections of shortest arcs by requiring them to coincide on the part from the first to the last of their common points. If we proceed in this manner in joining a system of points by shortest arcs in the converging metrics, and then choose a subsequence for which the entire system of shortest arcs converges to some system of shortest arcs in the limit metric, we obtain a simultaneous joining of the points by the limiting shortest arcs. Such arcs cannot have essential intersections with one another (Figure 44a) but they may have various kinds of one-sided tangencies to one another, as depicted in Figure 44b.

2. **Two estimates for polyhedral metrics.**

4. **Variation of the angle $\gamma$.** Suppose that in a polyhedral metric, in
a region homeomorphic to a disc, there is a reduced triangle $ABC$ without interior tails (see subsection 2 of Chapter III). From now on it will be assumed that this triangle is distinguished from the enclosing manifold. The measurement of distances and the laying off of shortest arcs will be carried out in the triangle itself.\footnote{We may suppose that we are taking into account also triangles with interior tails, but along with distinguishing them from the enveloping manifold we regard them as having been “cut” along entrant tails.}

We employ the following notations:

$$\begin{align*}
(x^+) &= \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \\
(x^-) &= \begin{cases} 0 & \text{if } x \geq 0, \\ |x| & \text{if } x < 0. \end{cases}
\end{align*}$$

The positive part $\omega^+$ of the curvature of the triangle will be the sum $\sum (2\pi - \theta_i)^+$, where $\theta_i$ is the complete angle around the vertices of the polyhedral metric lying inside the triangle, and the sum $\sum$ is extended over all those vertices. The negative part $\omega^-$ of the curvature is formed from the analogous sum $\sum (2\pi - \theta_i)^-$, and also from the sum $\sum (\phi_i - \pi)$ for all those interior points of the sides of the triangle at which the sector $\phi_i$ from the side of the triangle is larger than $\pi$. Moreover, in the presence of one or several exterior tails we add to $\omega^-$ the “portion of the negative curvature” which is concentrated at the branch point of the side, equal to the size of the sector from the side of the triangle. (Such a situation is illustrated in Figure 45 for the point $D$. The portion of the negative curvature of the triangle concentrated there is $\tilde{\alpha}$.)

Marking off points $X, Y$ on the sides $AB, AC$ of the distinguished triangle, we may as usual lay off distances $AX = x, AY = y, XY = z$ which define a certain plane triangle $T(AXY)$ with an angle $\gamma_T(X, Y)$ at the vertex $A$. (The subscript $T$ recalls the fact that the measurement of the distance $XY$ was carried out in the triangle $T$ itself.) If the triangle $T$ is convex, we have $\gamma_T = \gamma$, where $\gamma$ is the analogous angle obtained on measuring the distance $XY$ in the original manifold. Under the conditions indicated above the following theorem holds for the quantity $\gamma_T$.

\textbf{Theorem 1.} If the pairs of points $X_i, Y_i, i = 1, \cdots, r + 1$, form an in-
creasing sequence on the sides $AB$, $AC$ of the triangle $ABC$, i.e., if $x_i \leq x_{i+1}$, $y_i \leq y_{i+1}$, then the sum of the positive increments of the quantity $\gamma_T$ does not exceed $\bar{\omega}^-$:

$$\sum_{i=1}^r (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^+ \leq \bar{\omega}^-,$$

and the sum of the absolute values of the negative increments of $\gamma_T$ does not exceed $\omega^+$:

$$\sum_{i=1}^r (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^- \leq \omega^+.$$

It follows from the assertion of the theorem that for a continuous monotone variation of the points $X$ and $Y$ the function $\gamma_T(X(t), Y(t)) = \gamma_T(t)$ has bounded variation, the positive part of this variation not exceeding $\bar{\omega}^-$ and its negative part not exceeding $\omega^+$. 

**Remark.** If the angle $\alpha$ of the sector at the vertex $A$ in the triangle $ABC$ exceeds $\pi$, then inequality (2) may be strengthened, being replaced by the following.

$$\pi - \alpha + \sum_{i=1}^r (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^- \leq \omega^+.$$

Turning to the proof of the theorem, we may suppose that the side $BC$ is the unique shortest arc in the triangle. Otherwise it may be replaced by one further left, i.e., passing closer to the vertex $A$, and the resulting two-gon may be discarded. This does not affect the value of $\gamma_T$ and does not increase $\bar{\omega}^-$ or $\omega^+$. 

We begin the proof by marking off points $X$, $Y$ on the sides $AB$, $AC$ lying so close to $A$ that there are no points $X_i, Y_i$ between $X, Y$ and $A$, and that the triangle $AXY$ cut off from $ABC$ by the leftmost shortest arc $XY$ does not contain, either inside itself or on its sides $AX, AY$, vertices of the polyhedral metric, i.e., $AXY$ is isometric to a plane triangle. (If the angle $\bar{\alpha}$ of the sector at the vertex $A$ in the triangle $ABC$ were less than $\pi$, then the triangle $AXY$ would be a nondegenerate plane triangle, and for $\bar{\alpha} \geq \pi$ this triangle degenerates into a twice covered segment.) Thus we obtain a figure consisting of a plane triangle $T(AXY)$, of a two-gon $D(X,Y)$ bounded by a side $XY$ of the triangle $T(AXY)$ and the rightmost of the shortest arcs $XY$, and of the remaining portion $Q(XBCY)$.

We shall show that beginning with such a situation we may carry out a process of continuous deformation of the figure in question, in the course of which the following conditions will be observed.
a) The polyhedral character of the metric will be preserved. The lengths of the sides $AB$, $AC$, and $BC$ will be preserved. The position of the point $Y$ on the side $AC$ will be preserved. The whole figure in the large remains a reduced triangle without interior tails, and in particular the sides $AB$, $AC$, and $BC$ remain shortest arcs in the figure in question. Finally, the figure, as before, consists all the time of a certain plane triangle $T(AXY)$, a two-gon $D(XY)$, and the remainder $Q(XBCY)$.

b) The point $X$ moves continuously along $AB$ toward the side of vertex $B$, and the length of the side $XY$ of the plane triangle $T(AXY)$ in each position is equal to the distance $XY$ for the corresponding points $X, Y$ in the original triangle.

c) The piece $Q(XBCY)$ in each position is isometric to the piece of the original triangle excised by the rightmost of the shortest arcs joining in it the corresponding points $X$ and $Y$. The form of the two-gon $D(XY)$ may vary and does not necessarily coincide with that of the corresponding two-gon in the original triangle.

d) The total number of vertices of the polyhedral metric with curvatures distinct from zero does not increase. We are counting among these vertices also the point $Y$, which we have fixed, and the moving point $X$, whether or not these points have curvature distinct from zero.

e) The characteristics $\bar{\omega}^-, \omega^+$ of the triangle $ABC$ do not increase in the course of the deformation.

f) The angle at the vertex $A$ in the triangle $T(AXY)$ is simply $\gamma_T(X,Y)$. In passing from the original position of the point $X$ to any of the positions following it this angle undergoes an increment $\Delta\gamma_T$, which, if positive, does not exceed the decrease of $\bar{\omega}^-$, and if nonpositive does not exceed the decrease of $\omega^+$. The existence of the process just described leads to the proof of Theorem 1. Indeed, we may in turn shift the point $X$ in the indicated way, and then, changing the roles of $X$ and $Y$, move the point $Y$ and make these points go through all the positions of the $X_i$ and $Y_i$. Thus, from the fact that conditions e) and f) are preserved, the validity of Theorem 1 will follow.

In order to prove the existence of the necessary process, we shall establish that, allowing an arbitrary position of the points $X, Y$, we may push the point $X$ to a further position while preserving all the conditions a)–f). Along with the passage to the limit this makes it possible to shift the point $X$ to any point given in advance, in particular to the next point $X$. 
Thus we may suppose that the points $X, Y$ have already been shifted into a certain position $X_0, Y$ with conditions a)−f) preserved. In order to prove the existence of a further shift of the point $X$ from the position $X_0$, we consider five different possible cases.

**Case 1.** Suppose that for some point $X' \in (X_0, B]$, and therefore for all the points $X \in [X_0, X']$, the leftmost shortest arc $XY$ passes through $X_0$ (Figure 46a).

![Figure 46.](image)

In this case we may continuously shift the point $X$ from $X_0$ to $X'$, at the same time straightening out the quadrilateral $AXX_0Y$ into the plane triangle $AXY$ (Figure 46b). The distance from $A$ to any point of the side $XX_0Y$ does not decrease while this is being done, and thus the sides $AB$ and $AC$ remain shortest arcs.

If the triangle $T(AX_0Y)$ degenerates into a segment, then the angle $\gamma_T$ does not change. Otherwise, this angle increases. Suppose that $\xi$ is the angle of the sector at the vertex $X_0$ in the triangle $T$, and $\pi + \eta$ the angle at the vertex $X_0$ in the remaining part $D + Q$ (Figure 46a). On rectifying the quadrilateral the negative curvature at the point $X_0$ will be lost. This is equal in absolute value to $\xi + \eta$. The portion $\eta$ goes into the new vertex $X'_0$, and the portion $\xi$ is distributed between the increase in the absolute value of the negative curvature at the points $X$ and $Y$ and the change in $\gamma_T$. Thus the latter completely coincides with the total decrease in the quantity $\bar{\omega}^-$.

Evidently the remaining conditions a)−e) are preserved during the indicated process.

**Case 2.** Suppose that the conditions of Case 1 do not hold, i.e., that a leftmost shortest arc $XY$ for $X \in (X_0B]$ does not go through $X_0$. Suppose moreover that the two-gon $D(X_0Y)$ merges into a unique shortest arc $X_0Y$. 
In this case the shortest arcs \( XY \) converge to the side \( X_0Y \) of the triangle \( T \) as \( X \to X_0 \). Only vertices of the metric with negative curvature can lie on \( X_0Y \). Therefore, for \( X \) close to \( X_0 \) the shortest arcs \( XY \) pass close to \( X_0Y \) in a region not containing vertices with positive curvature, and therefore the shortest arcs \( XY \) themselves are unique. It is easy to see that when the point \( X \) moves from \( X_0 \) along \( X_0B \) the shortest arc \( XY \) originally changes continuously. Furthermore, it stretches along the shortest arc \( YX_0 \) on the portion from \( Y \) to the last of the vertices with negative curvature lying on \( YX_0 \), and on the last portion \( RX \) simply swings in a region isometric to the plane around the point \( R \) (Figure 47a).

![Figure 47.](image)

We shall move the point \( X \) from \( X_0 \), simultaneously straightening out the plane pentagon \( AX_0XRY \) into the plane triangle \( AXY \) (Figure 47b). During this rectification the distance of points of the portion \( XRY \) from the point \( A \) does not decrease, so that in the entire figure in the large the sides \( AB, AC \), remain shortest arcs. The decrease of the negative curvature at the points \( X_0 \) and \( R \) is distributed into an increment of the negative curvature at the points \( X \) and \( Y \) and a decrease of the angle \( \gamma_T \), which thus coincides with the total decrease of \( \omega^- \). The remaining conditions a) – e) are evidently satisfied.

Before turning to the further cases, we consider the possible structure of the two-gon \( D(X_0Y) \), if it does degenerate into the unique shortest arc \( X_0Y \), serving as a side of \( T \).

Evidently, in general, all the shortest arcs \( X_0Y \) moving in \( D \) divide \( D \) into a series of two-gons. The number of such pieces is finite, since in each of them there is at least one vertex with positive curvature. We consider that two-gon \( O'O'' \) (Figure 48) which entirely adjoins \( X_0Y \), and if there are several such two-gons, the one which lies closest to \( X_0 \).
two-gon $O'O''$ is cross-hatched in Figure 48.

There is at least one vertex $Q_1$ of positive curvature in $O'O''$. The point $Q_1$ can be joined to $A$ by a shortest arc which does not leave $T+O'O''$. If this shortest arc (see Figure 49) is not unique, then two

![Figure 48.](image1)

![Figure 49.](image2)

such shortest arcs form a two-gon inside which there is a new vertex $Q_2$ lying also inside $O'O''$. It may be joined to $A$ by a shortest arc which not only does not leave $T+O'O''$ but also does not issue from the two-gon $AQ_1$ in question. If that shortest arc is not unique, we turn to a new vertex $Q_3$ and so forth. Thus we arrive at some vertex $Q_i$ which is joined to $A$ by a shortest arc $AQ_i$ moving in $T+O'O''$, which is moreover unique among the the shortest arcs $AQ_i$ moving in a certain two-gon $AQ_{i-1}$ (or $T+O'O''$ if $i = 1$).

**Case 3.** Suppose that the shortest arc $AQ_i$ does not pass through any vertex with negative curvature. Suppose, moreover, that the triangle $T$ this time does not degenerate into a segment. In this case we extend the shortest arc $AQ_i$ beyond the point $Q_i$ to a segment $Q_iQ'$ which along with $AQ_i$ divides in two a sector around point $Q_i$. Any point $Q$ of the segment $Q_iQ'$ may be joined to $A$ by a shortest arc $AQ$ lying, as does the shortest arc $AQ_i$, in the two-gon $AQ_{i-1}$ (or in $T+O'O''$, if $i = 1$). Be-
cause of the uniqueness of $AQ_i$, as $Q \rightarrow Q_i$ the shortest arcs $AQ \rightarrow AQ_i$. Therefore, when $Q$ moves along a sufficiently small segment $(Q,Q')$, the shortest arc $AQ$ passes so close to $AQ_i$ that there are no other vertices between $AQ$ and $AQ_i$. In addition, there are exactly two shortest arcs $AQ$ and they encircle a single vertex $Q_i$ (Figure 50).

In each of the positions $Q \in (Q_i,Q')$ we may excise the indicated two-gon $AQ$ and paste its edges. In this process the distance $X_0Y$ decreases. Indeed, as is evident with the notations of Figure 50:

$$AG + GE > AE.$$ \(^2\)

Therefore, if one marks off a point $F$ so that $AGF = AE$, then $GE > GF$ and $X_0Y > X_0G + GF + EY$, i.e., after removing the triangle, when the points $E$ and $F$ coincide, there appears a shorter path $X_0G + GF + EY$ from $X_0$ to $Y$.

Suppose that when we excise the two-gon $AQ'$ the distance $X_0Y$ decreases by $2\varepsilon$. Then for any point $X \in (X_0,B]$ distant from $X_0$ by less than $\varepsilon$, the distance $XY$ after the excision of $AQ'$ turns out to be less than the original distance $XY$. Indeed

$$XY \leq X_0Y + X_0X = X_0Y - 2\varepsilon + XX_0 < X_0Y - XX_0 \leq XY.$$ 

In the original position there was no shortest arc $XY$ passing to the left of $Q = Q_i$, for otherwise the two-gon $O'O''$ would have been incorrectly chosen. Now after removing the two-gon $AQ'$ there appears such a shortest arc $XY$, which is shorter than $XY$. Therefore, for some position $Q = Q(X)$ on the interval between $Q_i$ and $Q'$ the removal of the two-gon $AQ$ leads to the appearance of a shortest arc $XY$ running to the left of $Q$, exactly equal in length to the original distance $XY$. This shortest arc $XY$ begins at the point $X$, enters $O'O''$, and on the portion from $O''$ to $Y$ it may be regarded as coinciding with the side $X_0Y$ of the triangle $T$. As $X \rightarrow X_0$ the shortest arc $XY$ converges to $X_0Y$, for otherwise we would have a new shortest arc $X_0Y$ going to the left of $Q_i$, which would contradict the choice of $O'O''$. By the same principle the shortest arcs $XY$ are unique for $X$ close to $X_0$ and change continuously as $X$ shifts from $X_0$, beginning with the position $X_0Y$.

On excising the two-gon $AQ$ the sides $AB$ and $AC$ remain shortest arcs. The decrease of the angle at the vertex $A$ is equal to the decrease in $\omega^+$. 

\(^2\)For this inequality to be strict we have supposed that the plane triangle $AX_0Y$ does not degenerate into a segment, so that $X_0Y$ does not pass through $A$ and the points $E$ and $G$ do not coincide.
arising from the replacement of the vertex $Q$, by the vertex $Q$. The shortest arc $\tilde{X}\tilde{Y}$ is obtained by a continuous shift from $X_0Y$. If at the point $X_0$ or at any point $R$ of the side $X_0Y$ has negative curvature, then plane figure $AX_0XRY$ would stop being a triangle. Then, simultaneously with the shift of $X$ and the change of the excised triangle $AQ$, we accomplish a rectification of the figure $AX_0XRY$ into a triangle $AXY$, as we already did in Case 2. The resulting decrease of the angle $A$ will be exactly equal to the loss in $\tilde{\omega}$. Thus the change in the large of the figure will satisfy all the conditions $a) - f$).

**Case 4.** The shortest arc $AQ_i$ passes through one or several vertices $R$ with negative curvature. The triangle $AX_0Y$ does not degenerate into a segment.

The shortest arc $AQ_i$, passing through a vertex with negative curvature, leaves on at least on one of the sides a sector angle larger than $\pi$. Suppose $R'$ is the vertex with negative curvature closest to $Q_i$ for which to the right of $AQ_i$ the sector is larger than $\pi$, and that $R''$ is the closest of the vertices where the sector to the left of $AQ_i$ is larger than $\pi$. It is possible that $R' = R''$. The case when on one of the sides there is no vertex at all with a sector larger than $\pi$ need not be considered; we may subsume it under the Case 3 already studied above. The difference is only that the two-gon $AQ$ will have the form depicted in Figure 51. But, as in

**Figure 51.**

Case 3, the removal of $AQ$ leads to a shortening of the distance $X_0Y$, making it possible to retain the discussion presented above. Thus in Case 4 we take only those situations for which the two-gons $AQ$, for $Q$ close to $Q_i$, merge into a shortest arc within the limits of the triangle $T$ and have their sides divergent only within the limits of the two-gon $O'O''$ (Figure 52).

**Figure 52.**
In Case 4 we shall shift the vertex $Q$, moving away the two-gon $AQ$, until one of the following positions occurs:

1) A new shortest arc $O'O''$ occurs;
2) One of the vertices $R', R''$ stops being a vertex with negative curvature;
3) On $R'Q$ or $R''Q$ there appears at least one new vertex with negative curvature;
4) The vertex $Q$ merges with one of the vertices with negative curvature;
5) The curve $AQ$ (after removal of the two-gon) stops being the unique shortest arc $AQ$ in the region in question (in $AQ_{-1}$ or $T+O'O''$).

In each of these cases we turn to the consideration of the newly obtained figures. In view of the finiteness of the number of vertices of the polyhedral metric all these cases can appear only finitely many times, and we sooner or later arrive at Case 3, which allows us to continue the shift of the point $X$.

We note that in the process of removing the two-gon $AQ$ under the hypotheses of Case 4 the sides $AB$, $AC$ remained shortest arcs, the plane triangle $AX_0Y$ and its angle $\gamma_T$ remained invariant, the curvatures $\omega^-$, $\omega^+$ did not increase, and all the conditions a)–f) remained fulfilled.

**Case 5.** Suppose finally that under the hypotheses of Case 3 or 4 the triangle $T$ degenerates into a segment.

Since Case 1 is excluded, degeneration of $T$ into the segment $AYX_0$ is impossible. If $T$ merges with the segment $AX_0Y$, then the shortest arc $AQ_i$ certainly passes through a vertex $Q'$ with negative curvature and the discussion may be carried out as in Case 4. It remains to consider the possibility of degeneration of $T$ into the segment $X_0AY$. If in addition the vertex $A$ lies outside the two-gon $O'O''$, the shortest arc $AQ_i$ passes through $O'$ or $O''$ and the discussion may be carried out as in Case 4. Suppose finally that the point $A$ belongs to the segment $O'O''$ (Figure 53). Then we shall distinguish two cases, depending on whether the angle $\tilde{a}$ of the sector at the vertex $A$ is larger than or equal to $\pi$.

If $\tilde{a} > \pi$, then removal of the two-gon $AQ$ for small shifts of $Q$ from $Q_i$ does not shorten $X_0Y$ and we are essentially back at Case 4.

If $\tilde{a} = \pi$, then removal of the two-gon $AQ$ is connected with the shortening of $X_0Y$ and we may proceed as in Case 3.

Thus it is proved that in any of the positions one may prolong the transformation of the figure $ABC$ while preserving the conditions a)–f).
At the same time, if the position of the point $X$ becomes unboundedly close to the position $X'$, one may pass to the limiting figure: the lengths of the sides of the triangle $T$ will converge to some quantities, the lengths of the sides of the limit triangle $T'$, and the piece $D + Q$ transforms by being monotonically shortened, at the expense of movement of the shortest arc $XY$ and the removal of two-gons $AQ$ with a finite number of displaced points $Q$. This piece also converges to some limit figure, dividing the rightmost of the shortest arcs $X'Y$ into pieces $D' + Q'$. In addition, the limit figure $T' + D' + Q'$ will satisfy all the conditions a) – f).

The possibility of prolonging the transformation from any position which may be reached and the possibility of passing to the limit enables us to shift the point $X$ in the indicated way and monotonically up to any position on the piece $X_0B$ of the side $AB$. As we have already noted, this proves Theorem 1.

Remarks. 1) Suppose that in the initial position the triangle $T$ reduces to a segment $XAY$, and the angle $\bar{\alpha}$ of the sector adjacent to the vertex $A$ in the figure $ABC$ is larger than $\pi$. After rectifying $ABC$ into a plane triangle $T(ABC)$ this angle will be not greater than $\pi$. We may observe that the initial decrease of this angle to $\pi$ occurred in the process of deformation of the triangle $ABC$ by removing the two-gon $AQ$ (Case 5). In Case 5 the decrease of $\bar{\alpha}$ is connected with the loss of the corresponding part of $\omega^+$. This makes it possible to strengthen the formulation of Theorem 1 and to pass from estimate (2) to estimate (3).

2) The proof of Theorem 1 can be somewhat simplified if the results of [65] are used. In fact, one may strike off the leftmost shortest arcs $X_iY_i$, and replace each of the regions homeomorphic to the disc into which these shortest arcs divide $T$ by a polygon on the cone. Then, to the resulting simpler polygons containing few vertices with $\omega^+ > 0$ and not containing any vertex with $\omega^- \neq 0$, we may apply the arguments presented above.

5. The oscillation of $\gamma$ farther from the vertex.

Theorem 2. Under the conditions of Theorem 1, if the positive part of the curvature of the triangle $ABC$ is concentrated near the vertex $A$,
then the decrease of $\gamma_T(X, Y)$ on pieces far from the vertex $A$ is small.

More precisely: for any $\Omega^+ \geq 0$, $R > 0$, $\varepsilon > 0$ given in advance, there exist $r > 0$ and $\delta > 0$ so small that in all cases when the positive part $\omega^+$ of the curvature of the triangle $ABC$ consists of the part $\omega^+ \leq \Omega^+$ concentrated in an $r$-neighborhood of the vertex $A$, and a remaining portion $\omega^+_2 \leq \delta$, the decrease of $\gamma_T(X, Y)$ on any segment outside $R$ of a neighborhood of the vertex $A$ does not exceed $\varepsilon$, i.e., for $R \leq AX_1 \leq AX_2$ and $R \leq AY_1 \leq AY_2$

(4) \[ \gamma_T(X_1, Y_1) - \gamma_T(X_2, Y_2) \leq \varepsilon. \]

The idea of the proof of Theorem 2 is the following. While carrying out the excisions of the two-gons $AQ$ enclosing the vertices $Q_i$ of positive curvature, we may move the vertices $Q_i$ away from $A$. This makes the curvature $\omega^+_1$ decrease and become very small. At the same time all the changes in the curvature $\omega^+_1$ can be realized in a small neighborhood of the point $A$ and do not affect in any essential way the quantities $\gamma_T(X_1, Y_1)$ and $\gamma_T(X_2, Y_2)$. Then it remains to apply Theorem 1 to the estimation of $\Delta \gamma_T$ on the piece $X_1, Y_1; X_2, Y_2$ in the metric measured in the indicated way.

In order to carry out this plan, we prove two lemmas.

**Lemma 8.** If a vertex $Q$ lying at a distance $r$ from $A$ and having positive curvature $\omega$ is removed to a larger distance $\rho$ by excision of a two-gon $AQ$ enclosing $Q$ and containing no other vertex of the metric, then the curvature $\omega$ decreases not less than $r/\rho$ times, taking on a value

(5) \[ \omega' \leq \omega \frac{r}{\rho}. \]

Indeed, in the general case the excised two-gon $AQ'$ has the form
depicted on Figure 54a. On Figure 54b, we depict in the form of a
development on the plane one of the two triangles \( AQQ' \) forming the
two-gon \( AQ' \). Evidently, in the notations of Figure 54,
\[
\omega' = \omega_1 + \omega_2, \quad r = r_2 + r_1, \quad \rho = \rho_2 + \rho_1, \quad r_2 < \rho_2.
\]

Using the decrease of the function \( \sin x/x \) on the interval \([0, \pi/2]\) and the sine law, we obtain
\[
\frac{\omega_2}{2} \leq \frac{\sin \omega_2}{\sin \omega_2} = \frac{r_2}{\rho_2} \leq \frac{r}{\rho}.
\]
Along with the analogous estimate for \( \omega_2 \) this yields inequality (5).

**Lemma 9.** If the shortest arc \( XY \) passes close to \( A \), then \( \gamma_T(X,Y) \) is
close to \( \pi \).

More precisely: if on a shortest arc joining the points \( X \) and \( Y \), which
are more distant from \( A \) than from \( R \), there is a point distant from \( A \)
by a distance \( \rho \leq (R/2) \sin^2 (\varepsilon/6) \), then
\[
(6) \quad \pi - \frac{\varepsilon}{3} \leq \gamma_T(X,Y) \leq \pi.
\]

Indeed, in the indicated case, for the distances \( x = AX, y = AY, z = XY \),
we have the inequality
\[
x + y - 2\rho \leq z \leq x + y.
\]

In a plane triangle with sides \( x, y, z \) we have for the angle \( \gamma_T \):
\[
\cos \gamma_T = \frac{x^2 + y^2 - z^2}{2xy},
\]
so that
\[
1 + \cos \gamma_T = \frac{(x + y - z)(x + y + z)}{2xy} \leq \frac{2\rho \cdot 2(x + y)}{2xy}
\]
or in other words
\[
\sin^2 \frac{\pi - \gamma_T}{2} \leq \rho \left( \frac{1}{x} + \frac{1}{y} \right) \leq 2 \frac{\rho}{R} \leq \sin^2 \frac{\varepsilon}{6}.
\]

Taking into account the fact that \( 0 \leq (\pi - \gamma_T)/2 \leq \pi/2 \), we therefore
obtain (6) and Lemma (9) is proved.

Now we proceed to the proof of Theorem 2.

Take \( \delta = \varepsilon/3, \quad \rho = (R/2) \sin^2 (\varepsilon/6), \quad \gamma = \rho \varepsilon/3 \Omega^+ \). Suppose that in an \( r \)-
neighborhood of the vertex \( A \) there are vertices of the metric with the
total positive curvature \( \omega_1^+ \leq \Omega^+ \). By excising two-gons \( AQ' \) we may shift
them off to the distance $\rho$. (If positive curvature exceeding $\pi$ is concentrated at a vertex, then initially in the shift such a vertex approaches $A$ and the curvature concentrated at it decreases. Then the curvature continues to decrease, and the vertex starts to move off from $A$.) After this, since all these vertices have moved off to a distance $\rho$, their total curvature, in accordance with Lemma 8, takes a value

$$\omega_{l}^{+} \leq \omega_{i}^{+} \frac{r}{\rho} \leq \frac{\varepsilon}{3}.$$

The shift of the vertices is connected with a change in the polyhedral metric, but none of these changes leaves the limits of the exterior boundary of the $\rho$-neighborhood of the vertex $A$. The distances of all points up to $A$ remain unchanged.

Suppose that $\gamma_{1} = \gamma_{T}(X_{1}, Y_{1}), \gamma_{2} = \gamma_{T}(X_{2}, Y_{2})$ in this initial metric, and that $\gamma_{1}', \gamma_{2}'$ are the corresponding angles in the transformed metric.

We apply Theorem 1 to the altered metric. We have:

$$\gamma_{1}' - \gamma_{2}' \leq \omega_{l}^{+} + \omega_{2}^{+} \leq \frac{2}{3} \varepsilon. \quad (7)$$

Now we consider the following three possibilities:

1. The shortest arcs $X_{1}Y_{1}, X_{2}Y_{2}$ in the changed metric do not touch the $\rho$-neighborhood of the point $A$. Then $\gamma_{1} = \gamma_{1}', \gamma_{2} = \gamma_{2}'$ and the inequality $\gamma_{1} - \gamma_{2} \leq \varepsilon$ mentioned in Theorem 2 follows from (7).

2. The shortest arc $X_{2}Y_{2}$ in the altered metric encounters the $\rho$-neighborhood of $A$. Then the shortest arc $X_{1}Y_{1}$ also encounters that neighborhood, since that shortest arc may be considered to pass to the left of $X_{2}Y_{2}$.

From Lemma 9 we have:

$$\pi - \frac{\varepsilon}{3} \leq \gamma_{1}' \leq \pi, \quad \pi - \frac{\varepsilon}{3} \leq \gamma_{2}' \leq \pi.$$

But the shortest arcs $X_{1}Y_{1}$ and $X_{2}Y_{2}$ in the unaltered metric either also encountered the $\rho$-neighborhood of $A$ or else passed through the unaltered zone, in which case they can only be longer than the newly appearing shortest arcs. Therefore

$$\pi - \frac{\varepsilon}{3} \leq \gamma_{1} \leq \pi, \quad \pi - \frac{\varepsilon}{3} \leq \gamma_{2} \leq \pi,$$

so that, further, for the absolute value of the difference in this case we have:

$$|\gamma_{1} - \gamma_{2}| < \varepsilon.$$
3. In the altered metric the shortest arc $X_1Y_1$ touches, and the shortest arc $X_2Y_2$ does not touch the $\rho$-neighborhood of $A$. Then $\gamma_2 = \gamma'_2$, and with an argument analogous to that of the preceding section we establish that $|\gamma_1 - \gamma'_1| \leq \varepsilon/3$.

Thus

$$\gamma_1 - \gamma_2 \leq |\gamma_1 - \gamma'_1| + \gamma'_1 - \gamma'_2 \leq \varepsilon.$$ 

Theorem 2 is proved.

Theorem 3. Under the hypotheses of Theorem 1, if the negative curvature of the triangle is concentrated near $A$, the increment of $\gamma_T(X, Y)$ on a segment distant from $A$ will be small.

More precisely: given any $\Omega^- \geq 0$, $R > 0$, $\varepsilon > 0$, there exists a pair $r > 0$, $\delta > 0$, so small that when the negative part of the curvature of the triangle $ABC$ consists of a part $\omega^- \leq \Omega^-$ and a remaining part $\omega^- \leq \delta$, the increment of $\gamma_T(X, Y)$ on any piece outside an $R$-neighborhood of the vertex $A$ does not exceed $\varepsilon$, i.e., for $A \leq AX_1 \leq AX_2$, $R \leq AY_1 \leq AY_2$

$$\gamma_T(X, Y) - \gamma_T(X_1, Y_1) \leq \varepsilon. \quad (8)$$

The proof of Theorem 3 is carried out on the same plan as that of Theorem 2. We need to remove vertices with negative curvature from the vicinity of $A$, to verify that their curvature decreases sufficiently fast, and that the altered metric does not essentially affect the difference $\gamma_2 - \gamma_1$, and then to apply Theorem 1.

Suppose that $P$ is a vertex with negative curvature of absolute value
Pass a shortest arc $AP$ through $A$ and prolong it by two segments $PP'$ each forming with $AP$ the angle $\pi$, as in Figure 55a. Divide the sector $\omega$ remaining between the two segments into $n$ pieces by $n-1$ further segments $PP'$. Then cut along the curve $AP$ and along all the segments $PP'$ and paste into the cut thus formed a plane figure as depicted in Figure 55b. In this pasting the vertex $P$ disappears and $n+1$ new vertices $P'$ appear in its place.

**Lemma 10.** If for the indicated pasting, there appear, in the place of the vertex $P$ distant by $r$ from $A$ and having negative curvature of absolute value $\omega$, $n+1$ vertices $P'$, each of which is distant by a greater distance $\rho$ equal to $AP + PP'$, then the absolute value $\omega'$ of the total curvatures of the vertices $P'$ satisfies the inequality

$$\omega' \leq \omega \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}.$$

For the proof we consider one of the triangles $APP'$, as in Figure 56: We have:

$$\frac{\sin \frac{\omega - \omega'}{2n}}{\sin \frac{\omega}{2n}} = \frac{\rho - r}{\rho},$$

so that

$$\sin \frac{\omega}{2n} - \sin \frac{\omega - \omega'}{2n} = \frac{r}{\rho} \sin \frac{\omega}{2n'},$$

$$2 \sin \frac{\omega}{4n} \cos \frac{2\omega - \omega'}{4n} = 2 \frac{r}{\rho} \sin \frac{\omega}{4n} \cos \frac{\omega}{4n'}.$$

$$\frac{\sin \frac{\omega'}{4n}}{\sin \frac{\omega}{4n}} = \frac{r}{\rho} \cdot \frac{\cos \frac{\omega}{4n}}{\cos \frac{2\omega - \omega'}{4n}} \leq \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}.$$

But then

$$\frac{\omega'}{\omega} \leq \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}$$

and Lemma 10 is proved.
We turn to the proof of Theorem 3. Suppose
\[ \delta = \frac{\varepsilon}{3}, \quad \rho = \frac{R}{2} \sin^2 \frac{\varepsilon}{6}, \quad r = \rho \frac{\varepsilon}{12\Omega}. \]

First, by excising two-gons, we move all vertices \( Q \) with positive curvature to the exterior boundary of the \((\rho/2)\)-neighborhood of the point \( A \). Under this the distances to \( A \) do not change. Then we join each of the vertices with negative curvature and lying in the \( r \)-neighborhood of the point \( A \) to \( A \) by a shortest arc. These shortest arcs will be unique. They can only "flow together" in the direction towards the vertex \( A \), since they are traced out in a zone where there do not remain any vertices with positive curvature, so that the formation of two-gons is impossible. Each of the shortest arcs \( AP \) will be prolonged to a distance \( \rho/2 \) from \( A \) by a bundle of segments \( PP' \) as described above. All the curves \( PP' + PA \) remain shortest arcs and do not intersect, in view of the absence of vertices with positive curvature. The number \( n \), for each of the bundles thus drawn, will be chosen so large that for the given vertex \( P \) the condition
\[ \sin \frac{\omega}{4n} \leq \frac{1}{2} \leq \cos \frac{\omega}{2n} \]
will be satisfied.

Then we carry out the pasting described above for each of the vertices \( P \). Vertices \( P \) with total negative curvature of absolute value \( \bar{\omega}^- \) are thus replaced by newly appearing vertices \( P' \) for which, according to Lemma 10, we will have the following estimate for the absolute value of their total negative curvature:
\[ \bar{\omega}^- \leq \bar{\omega}^- + 2r \min \frac{1}{\rho} \frac{1}{\cos \frac{\omega}{2n}} \leq \Omega^- \frac{\varepsilon}{6\Omega} - 2 = \frac{\varepsilon}{3}. \]

For the indicated pasting the distances of points to the vertex \( A \) do not decrease, so that \( AB \) and \( AC \) remain shortest arcs. Now the increase of any point \( M \) from \( A \) cannot be larger than \( \rho/2 \). Indeed, the shortest arc \( MA \) drawn before carrying out the pasting either did not touch the future cuts, in which case it has the same length also after the pasting, or else on some portion \( AM' \) it went along the curve of a cut (Figure 57a). In this case the shortest arc \( MA \), after pasting, is elongated by not more than \( 2x \sin(\omega'/4n) \), which does not exceed \( \rho/2 \), since \( x \leq \rho/2 \), and \( \sin (\omega'/4n) \leq 1/2 \) by the choice of \( n \).
After the indicated shift of the vertices $Q$ and the vertices $P$ all the changes in the metric do not go beyond the exterior boundary of the $\rho$-neighborhood of the point $A$, either in the original or in the altered metric.

After all this the proof of Theorem 3 is completed in just the same way as that of Theorem 2. We shall not repeat the argument here.

Remark. Putting Theorems 2 and 3 together, we may assert that if the basic part of the absolute curvature of a polyhedral triangle $ABC$ is concentrated near its vertex $A$, the oscillation of the quantity $\gamma_T(X,Y)$ far from $A$ turns out to be insignificant.

3. Existence of angles in the limit metric. In this section we prove the existence of an angle between any two shortest arcs in metrics admitting local approximation by polyhedral metrics for which the positive part of the curvature is uniformly bounded (see subsection 1). Thus we shall have established that in two-dimensional manifolds of bounded curvature any two shortest arcs which issue from the same point form a well defined angle. In subsection 13 of Chapter II this was proved only for shortest arcs forming a sector convex relative to the boundary.

6. Angle between shortest arcs. Suppose that the metric $\rho$ is the limit of uniformly converging polyhedral metrics $\rho_n$, all of which make $R$ into the same topological space, namely into a two-dimensional manifold. Let $G$ be a neighborhood homeomorphic to the closed disc of an arbitrary fixed point $O$. Suppose that in $G$ the positive parts $\omega_n^+$ of the curvatures of the metrics $\rho_n$ are bounded uniformly: $\omega_n^+(G) \leq C$. We shall consider $\omega_n^+$ to be a set function given on the Borel subsets of $G$.

It is well known that from an infinite system of completely additive functions, defined and uniformly bounded on the Borel sets of a com-
pact space it is possible to select a weakly converging sequence (see, for example, [1], Chapter III, p. 237). This makes it possible, retaining only some of the metrics \( \rho_n \), to suppose that the functions \( \omega_n^+ \) weakly converge to some completely additive set function \( \omega^+ \).

Suppose that \( U_r \) are neighborhoods of the point \( O \) homeomorphic to the disc minus the central point \( O \) and with radii not exceeding \( r \). As \( r \to 0 \) the sets \( U_r \) form a vanishing sequence. Therefore, in view of the complete additivity of the functions \( \omega^+ \) (see, for example, [1], Chapter II, p. 587),

\[
\lim_{r \to 0} \omega^+(U_r) = 0.
\]

We choose in \( U_r \) a set \( V_r \) homeomorphic to the circular annulus. It may be chosen so that the interior boundary of the ring is arbitrarily close to \( O \) and the exterior to the outer boundary of \( U_r \). Evidently \( \omega^+(U_r) \geq \omega^+(V_r) \). Moreover, in view of the weak convergence \( \omega_n^+ \to \omega^+ \) and the fact that \( V_r \) is closed,

\[
\omega^+(V_r) \geq \limsup_{n \to \infty} \omega_n^+(V_r)
\]

(see, for example, [1], Chapter III, p. 235).

Suppose the \( r \) is so small that \( \omega^+(U_r) < \varepsilon \). Then for each fixed \( V_r \) and for sufficiently large \( n \) we have \( \omega_n^+(V_r) < \varepsilon \).

From the above we may make the following assertion.

**Lemma 11.** There exists a subsequence of the metrics \( \rho_n \) such that for any \( \varepsilon > 0 \) there exists an arbitrarily small \( R \)-neighborhood of the point \( O \), and an arbitrarily small \( r \)-neighborhood of the point \( O \), chosen after the choice of \( R \), for which, beginning with sufficiently large \( n \), all the metrics \( \rho_n \) in the annulus between the \( R \)- and \( r \)-neighborhoods of the point \( O \) have positive curvature less than \( \varepsilon \).

Indeed, choose a sequence \( \rho_n \) for which \( \omega_n^+ \xrightarrow{w} \omega^+ \). We take \( R \) so small that \( \omega^+(U_{2R}-O) < \varepsilon \). Then we take an arbitrarily small \( r \) (\( 0 < r < R \)) and choose a ring-shaped region \( v \subset U_{2R} - O \) whose interior boundary lies inside the \( r \)-neighborhood of the point \( O \), and whose exterior boundary encloses the \( R \)-neighborhood of \( O \) but does not go outside of \( U_{2R} \). From what was said above, for sufficiently large \( n \) we will have \( \omega_n^+(V) < \varepsilon \).

Now it is easy to prove the following important assertion.

**Theorem 4.** In the limit metric any pair of shortest arcs issuing from one and the same point form a definite angle.
EXISTENCE OF ANGLES IN THE LIMIT METRIC

No additional hypotheses on the character of their relative positions is made.

In proving this theorem we shall not present precise \( \varepsilon \)-estimates, rather restricting ourselves to the essentials of the proof.

Suppose that the shortest arcs \( L \) and \( M \) issue from the point \( O \) and do not form a definite angle at that point. Then on these shortest arcs there exist two sequences of pairs of points \( X \) and \( Y \) converging to \( O \) for which the \( \gamma(X,Y) \) converge to limits differing by some \( \delta > 0 \).

Consider a very small \( R \)-neighborhood of the point \( O \). By the hypotheses, there exist on \( L \) and \( M \) pairs \( X_1, Y_1 \) and \( X_2, Y_2 \), for which \( OX_1 \leq OX_2 \), \( OY_1 \leq OY_2 \) and

\[
\gamma(X_1, Y_1) - \gamma(X_2, Y_2) \geq \frac{\delta}{2}.
\]

Suppose further that \( \rho_n \) is a polyhedral metric with a very large index. In it we draw shortest arcs \( OX_1, OX_2, OY_1, OY_2, X_1X_2, Y_1Y_2, X_2Y_2 \) without superfluous intersections, as for example in Figure 58. On the shortest arcs \( OX_2, OY_2 \) we mark points \( X'_1, Y'_1 \) distant from \( O \) by distances equal to \( OX_1 \) and \( OY_1 \).

From Lemma 11 we may suppose that the curvature \( \omega^+_n \) of the polyhedral metric \( \rho_n \) in the entire region in question is insignificantly small except possibly for some portion not exceeding a constant \( C \) independent of \( n \) and concentrated in the \( r \)-neighborhood of the point \( O \), where \( r \) is chosen to be insignificantly small in comparison with the distances \( OX_1, OY_1 \).

If we were to consider on the plane a triangle with the sides \( \rho(O,X_1), \rho(O,X_2), \rho(X_1X_2) \), then its angle at the vertex \( O \) would be equal to zero. In view of the uniform convergence \( \rho_n \to \rho \) we may suppose that the angle is close to zero also in distances measured in \( \rho_n \).

By Theorem 2, this last angle cannot grow strongly on the passage to the triangle \( OX_1, OX'_1, X_1X'_1 \). Therefore \( X_1X'_1 \) is very small in comparison with \( OX_1, OY_1 \). By a similar principle \( Y_1Y'_1 \) is small in comparison with \( OX_1, OY_1 \).

Now we have the following situation:

1) \( \gamma(X_1, Y_1) \approx \gamma_{\rho_n}(X_1, Y_1) \) because of the uniform convergence \( \rho_n \to \rho \);

2) \( \gamma_{\rho_n}(X_1, Y_1) \approx \gamma_{\rho_n}(X'_1, Y'_1) \) because of the smallness of \( X_1X'_1, Y_1Y'_1 \);
3) \( \gamma_n(X_1', Y_1') \) cannot greatly exceed \( \gamma_n(X_2, Y_2) \).

If the shortest arc \( X_1'Y_1' \) lies in the triangle \( X_2OY_2 \), then assertion (3) follows from Theorem 2. If now \( X_1'Y_1' \) goes outside this triangle, then on \( OX_2, OY_2 \), between \( X_1' \) and \( X_2 \) and between \( Y_1' \) and \( Y_2 \) there exist points \( X_1'' \) and \( Y_1'' \) joined by shortest arcs passing both inside and outside that triangle, and we obtain the same result on applying Theorem 2 first to \( X_1'Y_1' \) and \( X_1''Y_1'' \), and then to \( X_1''Y_1'' \) and \( X_2Y_2 \). Finally \( \gamma_n(X_2, Y_2) \approx \gamma(X_2, Y_2) \) in view of the uniform convergence \( \rho_n \to \rho \).

From this last and relations 1)–3) enumerated above, it follows that \( \gamma \) cannot substantially exceed \( \gamma(X_2, Y_2) \), in contradiction with hypothesis (9).

Thus Theorem 4 is proved.

4. Sector angle.

7. Sector angle. If two shortest arcs \( L_1, L_2 \) issue from the point \( O \) and on some initial segment have no essential intersections (not excluding the possibility that these shortest arcs overlap or touch), then it makes sense to say that these shortest arcs decompose a neighborhood of the point \( O \) into two sectors. Here the concept of sector is somewhat generalized; see subsection 11 of Chapter II.

A sector may in its turn be decomposed into pieces by a series of shortest arcs issuing from its vertex \( O \), going into the sector, and not having essential intersections. These shortest arcs, including the sides \( L_1 \) and \( L_2 \) of the sector itself, are ordered in their successive order around the point \( O \). The sector angle is the least upper bound of the sums of the angles between successive shortest arcs of each particular such decomposition of the sector, taken over all such decompositions. Below we shall verify that the sector angle is always finite in the spaces of interest to us. In the meantime we shall admit for it also infinite values.

The angle \( \alpha \) between the sides of a sector evidently does not exceed the angle \( \bar{\alpha} \) of the sector, since the sides of the sector themselves already form one of the possible decompositions referred to above.

**Lemma 12.** If the sector \( C(L_1, L_n) \) is divided into pieces by shortest arcs \( L_2, \ldots, L_{n-1} \), and if one of the subdividing shortest arcs forms a null angle with its successor, then such a shortest arc may be dropped from the curves of the subdivision, without changing the sum of the angles between successive shortest arcs.

If the shortest arcs \( L_1, L_2 \) and \( M \) issue from the point \( O \), and the shortest arc \( M \) has common points with \( L_1 \) and \( L_2 \) arbitrarily close to \( O \), then the shortest arcs \( L_1 \) and \( L_2 \) form a null angle.
Both assertions of Lemma 12 follow from the existence of angles and from the triangle inequality (Theorem 1 of Chapter II) for angles between three shortest arcs.

Lemma 12 makes it possible to supplement already existing subdivisions of a sector by any shortest arc lying in the sector. If in doing this the new shortest arc intersects, arbitrarily close to the vertex of the sector, one or several of the previous shortest arcs, then all of them along with the newly drawn shortest arc form pairwise null angles and may be replaced by a single new shortest arc without changing the sum of the angles between successive shortest arcs. But if the new shortest arc does not touch the other shortest arcs in a small neighborhood of \( O \), it simply supplements the decomposition, which can only increase the sum of the angles between successive shortest arcs of that decomposition.

The following important assertion therefore follows.

**Lemma 13.** In a two-dimensional manifold of bounded curvature, for sectors admitting decomposition into sectors convex relative to the boundary, the value of the sector angle in the sense of the definition of subsection 14 of Chapter II coincides with the value of that angle in the sense of the definition just given.

In fact, the new definition, extending the class of admissible decompositions, can only increase the value of the sector angle. But the possibility of supplementing any subdivision by the curves of the decomposition into sectors convex relative to the boundary shows that the increase of the value of the sector angle cannot take place.

The following theorem is proved analogously.

**Theorem 5.** Sector angles are additive.

More precisely, if the shortest arcs \( L_1, L_2, L_3 \) issue from the point \( O \) and do not have essential intersections near \( O \), then we may say that one of the sectors \( C_{13} \), bounded by the shortest arcs \( L_1, L_3 \), is made up of sectors \( C_{12} \) and \( C_{23} \). For the angles of these sectors \( \tilde{\alpha}_{13} = \tilde{\alpha}_{12} + \tilde{\alpha}_{23} \).

**Proof.** Each two decompositions of the sectors \( C_{12}, C_{23} \) forms a decompositions \( C_{13} \), so that \( \tilde{\alpha}_{13} \geq \tilde{\alpha}_{12} + \tilde{\alpha}_{23} \). Conversely, every decomposition \( C_{13} \) may be supplemented by the shortest arc \( L_2 \) without decreasing the sum of the angles between successive shortest arcs of the decomposition, so that \( \tilde{\alpha}_{13} \leq \tilde{\alpha}_{12} + \tilde{\alpha}_{23} \), whence \( \tilde{\alpha}_{13} = \tilde{\alpha}_{12} + \tilde{\alpha}_{23} \).

8. **Complete angle around a point.** The entire neighborhood of a point may be considered as a special sort of sector. We shall call its angle the
complete angle around the point and as a rule designate it by $\theta$. More precisely, $\theta$ is the upper bound of the sum $\sum_{i=1}^{n} \alpha_{i,i+1}$ of the angles between successive shortest arcs $L_1, L_2, \ldots, L_m, L_{n+1} = L_1$ of any decomposition of the neighborhood of this point into a finite number of sectors. Such decompositions exist. It suffices, for example, to take a single shortest arc issuing from $O$. The finiteness of $\theta$, which we shall verify later, is so far not assumed.

For points whose neighborhoods decompose into sectors convex relative to the boundary, the value of the complete angle in the sense of the definition of subsection 14 of Chapter II evidently coincides with the newly given definition.

It will be necessary for us to make an essential distinction between ordinary points, with $\theta > 0$, and singular points, with $\theta = 0$.

**Lemma 14.** At points where $\theta = 0$ any two shortest arcs form a null angle.

Indeed, if two shortest arcs intersect arbitrarily close to $O$, they form a null angle. But if they do not intersect near $O$, then they must also form in the present case a null angle. Otherwise a pair of these shortest arcs would give a decomposition of the neighborhood, reducing to a positive sum of angles, and we would have $\theta > 0$.

Evidently in this case, any sector at the vertex $O$ will have a null angle.

9. **Properties of the sector angle.**

**Theorem 6.** Suppose that from a point with complete angle $\theta$ there issue two shortest arcs, forming sectors $C_1, C_2$ with angles $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Then the following assertions hold.

1) The angle between shortest arcs is equal to the smallest of the three numbers $\tilde{\alpha}_1, \tilde{\alpha}_2, \pi$:

$$\alpha = \min [\tilde{\alpha}_1, \tilde{\alpha}_2, \pi].$$

2) For a convex sector $\tilde{\alpha}_1$

$$\alpha = \min [\tilde{\alpha}_1, \pi].$$

3)

$$\theta = \tilde{\alpha}_1 + \tilde{\alpha}_2.$$

4) If $\tilde{\alpha}_1 < \tilde{\alpha}_2$ (in particular if $\theta > 0$ and $\tilde{\alpha}_1 < \theta/2$), then the sector $C_1$ is convex. If $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \theta/2$ the sector $C_1$ may be either convex or not. If $\tilde{\alpha}_1 > \tilde{\alpha}_2 < \pi$ the sector $C_1$ is nonconvex.

5) If the supplementary sector $\tilde{\alpha}_2 > \pi$, then the sector $C_1$ is convex relative
to the boundary; if \( \alpha_2 < \pi \) the sector \( C_1 \) cannot be convex relative to the boundary; if \( \theta < \pi \) there are in general no sectors at the vertex in question which are convex relative to the boundary; if \( \alpha_2 = \pi \) the sector \( C_1 \) may or may not have the property of convexity relative to the boundary.

**Proof.** 1. If close to the vertex \( O \) the sides of the sector prolong one another and form a single shortest arc, then \( \alpha = \pi, \alpha_1 \geq \pi, \alpha_2 \geq \pi \) and equation (10) holds. Otherwise shortest arcs \( XY \) that are very close to \( O \) and join points on the sides of the sector avoid \( O \) and lie either in the sector \( C_1 \) or in the sector \( C_2 \). Suppose for \( X, Y \) arbitrarily close to \( O \) that there is such a shortest arc in the sector \( C_1 \). Then by Theorem 3 of Chapter II we have \( \alpha = \alpha_1 \). But always \( \alpha \leq \pi, \alpha \leq \alpha_2 \), i.e., again (10) is true.

2. Equation (11) is proved in the same way. In this case, \( XY \) passes in \( C_1 \).

3. The equation \( \theta = \alpha_1 + \alpha_2 \) follows from the additivity of the sector angle.

4. If \( \alpha_1 < \alpha_2 \), then from (10) we have \( \alpha < \alpha_2 \) and the shortest arcs \( XY \) close to \( O \) cannot lie in \( C_2 \) since the sector \( C_1 \) is convex. If \( \alpha_2 < \alpha_1 \) and \( \alpha_2 < \pi \), then, from (10), we have \( \alpha < \alpha_1 \) and the shortest arcs \( XY \), if sufficiently close to \( O \), cannot lie in \( C_1 \). Thus the sector \( C_1 \) is nonconvex. Examples for both convex and nonconvex sectors with \( \alpha_1 = \alpha_2 \) are easily constructed on surfaces close to the cone.

5. The proof of the last (fifth) assertion of Theorem 6 is given below in subsection 13.

10. **Density of shortest arcs. Finiteness of the sector angle.**

**Theorem 7.** A neighborhood of any point \( O \) may be divided into sectors by a finite number of shortest arcs issuing from that point in such a way that they will successively form with one another angles less than any given \( \varepsilon > 0 \).

**Proof.** In the case \( \theta = 0 \) only one shortest arc is needed. Suppose that \( \theta > 0 \). We choose a very small neighborhood \( U \) of the point \( O \) homeomorphic to the disc. From Lemma 11 one may assume that \( U \) is so small, and the sequence of polyhedral metrics \( \rho_n \rightarrow \rho \) is specialized in such a way, that for large \( n \) the fundamental part \( \omega^+_n(U) \) of the positive curvature in the metrics \( \rho_n \) is concentrated in an insignificantly small neighborhood of the point \( O \).

In \( U \) we carry out the following construction. We encircle the point \( O \) with a simple closed curve \( \Gamma \). We subdivide this by points \( X_i \), into
pieces so small that, first, the maximum $d$ of their diameters in the metric $\rho$ is very small with respect to the distance $r$ from the point $O$ to the curve $\Gamma$ and, second, any shortest arcs in the metric $\rho$, successively joining the endpoints of these segments, form a curve enclosing $O$. In view of the uniform convergence $\rho_n \to \rho$, these conditions will also be satisfied in the metrics $\rho_n$ with sufficiently large index.

In the polyhedral metric $\rho_n$ we successively join these division points by shortest arcs, avoiding superfluous intersections in the process. We obtain a closed broken curve enclosing the point $O$. If successive links of this polygonal curve have a common initial segment, we discard it. If the polygonal curve has self-intersections even after this, we reject the loops thus formed. We arrive at a simple closed polygonal curve enclosing $O$. In the process the total number of vertices does not increase and all the vertices will lie at a distance not less than $r - d - \delta(n)$, where $\delta(n) > 0$ is arbitrarily small for large $n$.

Now we join the vertices of the resulting polygonal curve to the point $O$. If the next shortest arc $AO$ leaves the limits of the broken polygonal curve and again intersects it at the point $M$, we reject the piece $AM$ of our curve, replacing it by the piece $AM$ of the shortest arc $AO$. After finishing this joining operation we obtain a finite system of reduced triangles $OAB$, of which there are no more than the number of points $X_i$ in the original subdivision of $\Gamma$. The sectors at the vertices $O$ of these triangles, successively adjoining one another, fill out a neighborhood of the point $O$. The sides $OA$, $OB$ of these triangles cannot turn out to be much smaller than the distance from $O$ to the curve $\Gamma$ in the original metric $\rho$, and the sides $AB$ will be very small in comparison with $OA$ and $OB$.

This construction may be repeated in each of the metrics $\rho_n$, and then a subsequence chosen for which the resulting curves of the subdivision of the neighborhood of $O$ converge to some subdivision of a neighborhood of the point $O$ by shortest arcs in the metric $\rho$.

We assert that under appropriate choice of the neighborhood $U$ and sufficient fineness of the subdivision of the curve $\Gamma$, the resulting subdivision of the neighborhood of $O$ by shortest arcs will satisfy the requirements of Theorem 7. In fact, if the angle $\alpha$ between successive shortest arcs were large ($\geq \varepsilon$), then the angle $\gamma$ would also be large in certain positions of the points $X, Y$ on that shortest arc. But then the angle $\gamma'$ for the corresponding points $X_n, Y_n$ on the sides converging to this shortest arc of the
triangle $OAB$ in some polyhedral metric $\rho_n$ would also be large. But then *a fortiori* the angle $\gamma_T^n$ obtained by measuring the distance inside the triangle $OAB$ (in the metric $\rho_n$) would also be large. Finally, the quantity $\gamma_T^n(X_n, Y_n)$, in view of Theorem 2, cannot be noticeably decreased by shifting the points $X_n, Y_n$ into the position $A, B$, since the positive curvature $\omega_n^+$ of the polyhedral metric $\rho_n$ in the triangle $OAB$ is very small, with the exclusion possibly of a bounded quantity concentrated in a neighborhood of the point $O$ insignificantly small in comparison to $OX_n, OY_n$. This last may be accomplished by a sufficiently large choice of $n$. The quantity $\gamma_T^n(A, B)$ is small, so that the side $AB$ is very small in comparison with $OA$ and $OB$. Thus the quantity $\alpha$ is small and Theorem 7 is proved.

**Remark.** The required subdivision is based on a system of shortest arcs which are limiting for systems of arcs subdividing the neighborhood of the same point $O$ into sectors of reduced triangles in certain metrics $\rho_n$.

A number of corollaries follow from Theorem 7.

**Theorem 8.** *If $\theta > 0$, the neighborhood of the point may be divided into arbitrarily small convex sectors.*

In addition the subdivision may be accomplished by shortest arcs which are limits of shortest arcs in approximating polyhedral metrics.

Indeed, from Theorem 7, there exists a subdivision with angles between successive shortest arcs smaller than $\theta/2$. From item 4) of Theorem 6, these sectors are convex.

In this case it is not only true that in a sufficiently small neighborhood of the point $O$, for any pair of points $X, Y$, on the opposite sides of each sector of the subdivision, there exists a shortest arc $XY$ passing in that sector, *but also that there does not exist any one shortest arc $XY$ which encloses $O$ from outside.* Indeed, if there were such shortest arcs arbitrarily close to $O$, then we would arrive at a contradiction with the condition $\bar{\alpha} < \theta/2$.

**Remark.** In the case $\theta = 0$ it may turn out in general to be impossible to subdivide a neighborhood of a point into convex sectors. One may obtain

*Figure 59.*
an example of this by an appropriate choice of the dimensions of the
twice-covered plane figure depicted in Figure 59. Here the protuberances
converge to the point $O$.

**Lemma 15.** If $\theta > 0$, then the equation $\theta = \sum \alpha_i$ holds for every subdivision
by shortest arcs, successively forming the angles $\alpha_i \leq \min(\pi, \theta/2)$.

Indeed, in this case all the sectors of the decomposition are convex.
By Theorem 6, $\alpha_i = \bar{\alpha}_i$ for each of them. But by the additivity of the sector
angles we have $\sum \bar{\alpha}_i = \theta$, so that $\theta = \sum \alpha_i$.

**Theorem 9.** The complete angle $\theta$ around a point is always finite.

This follows from Lemma 15 and the existence of the necessary subdivision
for $\theta > 0$.

**Corollary.** A sector angle is always finite.

5. **Boundedness of the absolute curvature of the approximating metric.**

In this section we prove that in the case of interest to us, the approxi-
mating polyhedral metrics have a local uniform bound not only for the
positive parts of the curvature, but also for the negative parts and thus
for the absolute curvatures.

11. **Polygonal neighborhood of a point.**

**Lemma 16.** Every point has a neighborhood $P$ of radius less than any
given $\varepsilon > 0$, where $P$ is a polygon homeomorphic to a disc and with a
perimeter less than $\varepsilon$. Moreover, for sufficiently large $n$ the point $O$ has
analogous neighborhoods $P_n$ in the metrics $\rho_n$. The number of vertices of
each of the polygons is bounded independently of $n$, and the point $O$ lies
in all the $P_n$ and in $P$ along with some fixed neighborhood $U$.

We shall carry out the proof separately for the cases $\theta > 0$ and $\theta = 0$.

Suppose that $\theta \neq 0$. We decompose a neighborhood of the point $O$ by
shortest arcs in the metric $\rho$ into sectors with angles $\alpha_i = \bar{\alpha}_i < \min(\pi, \theta/2)$.
This is possible from Theorem 7. Moreover, we may suppose that the
indicated shortest arcs are limits of sides of triangles $OA_iB_i$ enclosing the
point $O$ in the metrics $\rho_n$. The number of these triangles is bounded
by some number $N$ not depending on $n$.

The sector of this subdivision will be convex. On each of the shortest
arcs of the subdivision we mark a point $X_i$ at a distance $\delta < \varepsilon/2N$ from
$O$. Since $\delta$ is small and the sector convex, we may successively join
these points by shortest arcs $X_iX_{i+1}$ passing through the corresponding
sectors. We obtain a polygon $P$. 
It may occur that adjacent sides of \( P \), issuing from the vertex \( X_i \), coincide on some initial segment. In this case we move \( X_i \) to the branch point of these shortest arcs. We obtain a polygon \( P \) homeomorphic to the disc.

Note that, generally speaking the points \( X_i, X_{i+1} \) can be joined by various shortest arcs, including some which leave the sector \( X_1 OX_{i+1} \), but for sufficiently small \( \delta \) these shortest arcs, as has already been observed in the proof of Theorem 8, cannot enclose the vertex \( O \) of the sector \( X_1 OX_{i+1} \), since that would contradict the condition \( \alpha_i < \theta/2 \). They cannot pass through \( O \) either, since \( \alpha_i < \pi \). Moreover, they cannot touch any very small (in comparison with \( \delta \) ), neighborhood \( U \) of the point \( O \), since \( \alpha_i \) is strictly less than \( \pi \).

Suppose that \( X_i^* \) are the points on the sides of the triangles \( OA_iB_i \), converging to the points \( X_i \). From what has just been said, beginning with some \( n \) the shortest arcs \( X_i^*X_{i+1}^* \) in the metrics \( \rho_n \) cannot enclose the vertex \( O \) or touch the closed neighborhood \( \bar{U} \). Otherwise there would be a limiting shortest arc \( X_iX_{i+1} \) with the same property. This makes it possible, for sufficiently large \( n \), to draw shortest arcs \( X_i^*X_{i+1}^* \) in the triangles \( OA_iB_i \). The sectors of \( O \) of these triangles for large \( n \) are so to speak “convex away from \( O \).” Laying off in this way the shortest arcs \( X_i^*X_{i+1}^* \), we obtain in \( \rho_n \) a polygon \( P_n \). It will have a small perimeter, and a bounded number of vertices, and it will contain the point \( O \) and the entire neighborhood \( \bar{U} \). If adjacent sides of \( P_n \), issuing from \( X_i^* \), have a common origin, then we move \( X_i^* \) to the branch point of these sides. Thus we may make all the \( P_n \) homeomorphic to the disc.

Now suppose that \( \theta = 0 \). Encircle the point \( O \) and a neighborhood of diameter \( d < \varepsilon/2 \) by a closed curve \( \Gamma \). Since there does not exist any shortest arc through the point \( O \), no shortest arc joining the points of the curve \( \Gamma \) can either pass through \( O \) or touch a certain neighborhood \( \bar{U} \) of the point \( O \). Therefore also in the metrics \( \rho_n \) with \( n \) sufficiently large, no shortest arc joining two points of the curve \( \Gamma \) can encounter \( \bar{U} \). This makes it possible both in the metric \( \rho \) and in the metrics \( \rho_n \), to construct, for larger \( n \), two-gons which will lie in an \( \varepsilon \)-neighborhood of \( O \), have perimeters less than \( \varepsilon \), and enclose the point \( O \) along with some neighborhood \( U \) of \( O \). These two-gons will play the roles of \( P \) and \( P_n \).

Remark. The polygons \( P, P_n \) may be regarded as convex. It is sufficient to consider shortest loops, enclosing the originally chosen polygons \( P \) and \( P_n \), as was done in subsection 4 of Chapter III.
Theorem 10. In some neighborhood $U$ of the point $O$ both the positive and negative curvatures of the converging polyhedral metrics $\rho_n$ are bounded uniformly.

Proof. Suppose that $U$ is a neighborhood of the type indicated in Lemma 16. For each $n$ the neighborhood $U$ is contained in the polygon $P_n$. The number of sides $m$ of the polygon $P_n$ is bounded by a number $N$ not depending on $n$. Moreover, $\omega_n^+(P_n) \leq C$.

The polygon $P_n$ itself is homeomorphic to the disc and may be divided into plane triangles (Figure 60). As in subsection 9 of Chapter III, we find from the Euler theorem that

\[
\sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) + \left[ \sum \xi - (m - 2)\pi \right] = 0.
\]

Here the first sum is extended over all vertices of the polyhedral metric which are interior for $P_n$ and is equal to $\sum (\bar{\omega}_n^- - \omega_n^+)$ for all these vertices. The second sum is extended over all the vertices of the polyhedral metric lying within the sides of $P_n$. It represents that portion of $\bar{\omega}_n^- (P_n)$ which was not taken into account in the previous sum. Finally, the third term is the difference between the sum of the angles at the vertices of $P_n$ and the sum of the angles of a plane $n$-gon.

From (13) we immediately obtain the estimate

\[
\bar{\omega}_n^- (P_n) = \omega_n^+ (P_n) + (m - 2)\pi - \sum \xi \leq C + (N - 2)\pi,
\]

which proves Theorem 10.

6. Further information on angles.

12. Uniform closeness of $\gamma$ to $\alpha$. The boundedness of $\omega_n^+$ made it possible for us to choose a subsequence of metrics $\rho_n$ for which the $\omega_n^+$, as set functions, converged weakly to some limit function. The boundedness of $\bar{\omega}_n^-$, established in Theorem 10, makes it possible to specialize this subsequence in such a way that the functions $\bar{\omega}_n^-$ will also converge weakly to some limit function.

In Lemma 11 we have proved the possibility of choosing a neighborhood

---

Footnote: As in a triangle, we include in the quantity $\bar{\omega}^-(P)$ the absolute value of the negative rotations from the side of $P$ at the vertices of the polyhedral metric which lie on the sides of $P$ and do not coincide with the vertices of $P$. 
of the point $O$ so small that for the selected metrics $\rho_n$ their positive curvature will be insignificant except possibly for a bounded part of the curvature localized in an insignificantly small (in comparison with the neighborhood chosen before) region of the point $O$. Now we may assert the same result also for the negative part of the curvature.

In the course of the proof of Theorem 4, in using Lemma 11 and Theorem 2, we established that for any point $O$ and any $\varepsilon > 0$ there exists an arbitrarily small neighborhood $G$ of that point such that within the limits of this neighborhood, for two shortest arcs issuing from $O$ and two pairs of points $X_1, Y_1$ and $X_2, Y_2$ on them, with $OX_1 \leq OX_2$ and $OY_1 \leq OY_2$, the decrease in the angle $\gamma$ on passing to the more distant pair of points does not exceed $\varepsilon$:

$$\gamma(X_1, Y_1) - \gamma(X_2, Y_2) \leq \varepsilon.$$

Arguing on the analogy of Lemma 11 and Theorem 3, we can now prove, for some neighborhood $G$ the corresponding estimate of the possible increment in $\gamma$.

$$\gamma(X_2, Y_2) - \gamma(X_1, Y_1) \leq \varepsilon.$$

Putting these results together, we arrive at the following theorem.

**Theorem 11.** For any point $O$ and any $\varepsilon > 0$, there exists a neighborhood $G(\varepsilon, O)$ such that, for any pair of shortest arcs issuing from $O$ and any points $X, Y$ lying in $G$, on these shortest arcs the inequality $|\gamma(X, Y) - \alpha| \leq \varepsilon$ holds, where $\alpha$ is the angle between the shortest arcs.

We note a special consequence of Theorem 11.

**Lemma 17.** If at the point $O$ the complete angle $\theta = 0$ and if $L$ is a shortest arc issuing from $O$, then for any $\varepsilon > 0$ there exists an arbitrarily small neighborhood $G$ of the point $O$ such that through each point $X$ on $L$ in the region $G$ we may pass a loop enclosing $O$ whose length is less than $\varepsilon r$, where $r$ is the distance from $O$ to $X$.

**Proof.** We choose a neighborhood $G$ so small that in it $|\gamma - \alpha| < \varepsilon/2$ throughout. This we do by Theorem 11. Suppose that $X \in L \cap G$. Since $X$ lies on a shortest arc issuing from $O$ and which extends beyond $X$, $X$ is on the exterior boundary of the set of points distant from $O$ by a distance not larger than $r$. After choice of a very small $\delta > 0$, we may pass a simple closed curve $\Gamma$ enclosing $O$, all of whose points are distant from $O$ by a distance differing from $r$ by not more than $\delta$.

Joining $X$ to a point $Y$ running along the curve $\Gamma$, we find a two-gon
metrics admitting approximation by polyhedral metrics

$D(X,Y)$ enclosing $O$. We join its vertices to $O$ by shortest arcs $L_1, L_2$ lying in $D$. Since $\theta = 0$, the angle between these shortest arcs is equal to zero as well, so that $\gamma(X,Y) < \varepsilon/2$. But the distance $OX = r$, and the distance $OY$ differs from $r$ by less than $\delta$, so that the length of the side $XY$ of the two-gon $D$ is very small in comparison with $r$. The contour of the two-gon $D$, thus satisfies the requirements of Lemma 17.

13. Angle from the side of the sector. To each sector, besides the angle $\alpha$ between its sides and the angle $\bar{\alpha}$ of the sector itself, one may also attach the concept of the angle between sides measured in the sector itself. We understand by this the limit $\hat{\alpha}$ of the angles $\hat{f}(X,Y)$, constructed with respect to the distances $OX, OY$ and the distance $XY$ measured in the sector itself, i.e. with respect to the shortest of the curves $XY$ which lie in the sector.

Theorem 12. The angle $\hat{\alpha}$ between sides $M_1$ and $M_2$ of a sector, measured in the sector itself, always exists and is equal to

$$\hat{\alpha} = \min \{\pi, \bar{\alpha}\},$$

where $\bar{\alpha}$ is the sector angle.

Proof. 1. Suppose that $\theta > 0$ at the point in question, and that for the sector of interest to us $\bar{\alpha} < \min(\pi, \theta/2)$. Then, from Theorem 6, the sector is convex. In this case $\hat{f} = \gamma$, so that the limit $\hat{\alpha}$ exists and $\hat{\alpha} = \min \{\pi, \bar{\alpha}\}$. 2. Now suppose that $\theta > 0$ and $\bar{\alpha} \geq \min \{\pi, \theta/2\}$. We decompose a neighborhood of the point $O$ into fine sectors by shortest arcs whose successive angles between shortest arcs are larger than zero but less than $(1/2) \min \{\pi, \theta/2\}$. We supplement this decomposition with the sides $M_1, M_2$ of the original sector. If the latter intersects shortest arcs of the subdivision arbitrarily close to $O$, we drop the intersecting shortest arcs. We obtain a decomposition of our sector $M_1OM_2$ by a series of shortest arcs $L_i$ successively forming, including the sides of the sector, angles less than $\min(\pi, \theta/2)$. All the sectors of this decomposition are convex and the angles of these sectors are equal to the angles between their sides.

We now distinguish the whole sector $M_1OM_2$ and consider the intrinsic metric induced by this selection. Evidently, from the general theorem on upper angles, we have

$$\limsup \hat{f} \leq \sum \hat{\alpha}_i = \sum \alpha_i = \sum \bar{\alpha}_i = \bar{\alpha}.$$  

If the shortest arc $\overline{XY}$ in the sector $M_1OM_2$ also goes through $O$ for some $X$ and $Y$, then for $X, Y$ close to $O$, we always have $\hat{f} = \pi$, so that
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\[ \lim \hat{\alpha} \] exists and is equal to \( \pi \). Moreover, from (15) it follows that also \( \bar{\alpha} \geq \pi \), so that equation (14) is satisfied in this case.

If now all the shortest arcs \( XY \) avoid \( O \), then in analogy with equation (4') in Chapter II we obtain

\[
(16) \quad \lim \inf \hat{\alpha} \leq \sum \alpha_i = \bar{\alpha}.
\]

Moreover, in this case \( \lim \sup \hat{\alpha} \leq \pi \).

Along with inequality (15), inequality (16) obtained above indicates the existence of the limit \( \lim \hat{\alpha} = \bar{\alpha} \), and again equation (14) is valid.

3. Suppose finally that \( \theta = 0 \). In this case \( \bar{\alpha} = 0 \). It remains for us to show that \( \lim \hat{\alpha} = 0 \).

Suppose that \( X \) and \( Y \) are points on the sides \( M_1, M_2 \) of the sector in question, very close to \( O \), and suppose for definiteness that \( OX \leq OY \). Using Lemma 17, through the point \( X \) we draw a curve enclosing \( O \) whose length is small in comparison with \( OX \). On the shortest arc \( M_2 \) we then find a point \( Y' \), joined to \( X \) in the sector in question by a curve \( XY' \) which is very short in comparison with the distances \( OX \) and \( OY' \).

If we were to develop the triangle \( Y'OY \) on the plane, its angle at the vertex \( O \) would be equal to zero. This angle changes very little if we replace the side \( OY' \) by the side \( OX \) close to it in length, and the side \( Y'Y \) by a shortest arc in the sector of the curve \( XY \), which cannot be longer than \( YY' + Y'X \). Both changes in length are small in comparison with \( OX \) and \( OY' \). Therefore the angle \( \hat{\alpha}(X,Y) \) will also be very small and \( \lim \hat{\alpha} = 0 \).

Theorem 12 is completely proved.

Remarks. 1) From Theorem 12 it follows that on distinguishing a sector from the enclosing space the value of the sector angle in the induced intrinsic metric remains equal to the preceding.

2) The last of the assertions of Theorem 6 follows from Theorem 12. Suppose that \( \bar{\alpha}_2 > \pi \). If the sector \( C_1 \) were not convex relative to the boundary, then in \( C_2 \), for \( X \) and \( Y \) arbitrarily close to \( O \), there would be a path \( XY \) shorter than \( XO + OY \). But then we would have \( \bar{\alpha}_2 \leq \pi \), which from (14) contradicts the hypothesis \( \bar{\alpha}_2 > \pi \). Suppose that \( \bar{\alpha}_2 < \pi \). Then \( \bar{\alpha}_2 < \pi \) and there is a path \( XY \) in \( C_2 \) shorter than \( XO + OY \). Hence \( C_1 \) cannot be convex relative to the boundary. Examples of \( C_1 \) which are both convex and nonconvex relative to the boundary may be constructed for surfaces close to conical surfaces when \( \bar{\alpha}_2 = \pi \).
14. **Angle in the strong sense.**

**Theorem 13.** If the shortest arcs $OA$, $OB$ issuing from the same point $O$ lie along with all the shortest arcs joining their endpoints in a neighborhood of the point $O$, then between the segments $OA$ and $OB$ of these shortest arcs there exists an angle in the strong sense.$^4$

For the proof we have to consider sequences $\gamma(X_n, Y_n)$ for $X_n \to O$, $X_n \in (OA)$; $Y_n \to Y_0$, $Y_n \in (OB)$, $Y_0 \in [OB]$ under the condition that there exist shortest arcs $X_nY_n$ converging to the piece $OY_0$ of the side $[OB]$. The ordinary angle, i.e., $\lim \gamma(X_n, Y_n)$ as $X_n, Y_n \to O$, exists. Suppose that its value is equal to $\alpha$. It remains to be verified that the same limit will be obtained for any of the sequences indicated above with $Y_0 \approx O$.

We consider two cases.

1. Suppose that the shortest arcs $OA$ and $OB$ have points in common arbitrarily close to $O$. Then for $X_n$ sufficiently close to $O$ and $Y_n$ close to $Y_0$, the exact equation $\gamma(X_n, Y_n) = 0$ is valid and the limit of $\gamma$ evidently exists and is equal to zero. In this case $\alpha$ is also equal to zero.

2. Now suppose that the shortest arcs $OA$ and $OB$ divide the neighborhood of $O$ into two sectors.

![Figure 61](image1.png) ![Figure 62](image2.png)

On both sides of $OB$ and close to $O$ choose points $B'$ and $B''$, very close to a point $C$ of the shortest arc $OB$ distinct from $O$ (Figure 61). The points $B'$ and $B''$ are then joined to $O$ by shortest arcs $OB'$, $OB''$, without creating essential intersections with $OA$ and $OB$. By Theorem 11, these shortest arcs form very small angles with $OB$.

---

$^4$ See subsection 5 of Chapter II.
Now suppose that \( n \) is so large that the shortest arc \( X_nY_n \), passing close to \( OB \), intersects \( OB' \) or \( OB'' \) at some point \( Z \). We suppose that \( Z \neq O \), since if \( Z \) coincides with \( O \) the discussion is only simplified: in this case \( \gamma(X_n, Y_n) = \alpha = \pi \).

We develop on the plane the triangles \( OZX_n \) and \( OZY_n \), and we adjoin them to one another along the side \( OZ \) as in Figure 62. The angle \( \alpha_0 \) in the plane triangle \( OX_n Z \) is very close to the angle \( AOB' \), from Theorem 11. \( AOB' \) is, in view of the smallness of the angle \( BOB' \), close to the angle \( \alpha = \triangle AOB \). Straightening out the side \( X_nZY_n \) into the plane quadrilateral \( OY_nZX_n \), we establish that

\[
\gamma(X_n, Y_n) \geq \alpha_0 \simeq \alpha.
\]

Since the whole construction can be carried out in such a way that this last approximate equation is arbitrarily exact for large \( n \), we have

\[
\liminf_{n \to \infty} \gamma(X_n, Y_n) \geq \alpha.
\]

But from the general theorem on the upper angle (Theorem 4 of Chapter II) it follows that

\[
\limsup_{n \to \infty} \gamma(X_n, Y_n) = \alpha \simeq \alpha.
\]

Along with the foregoing this indicates the existence of the limit of the angle \( \gamma \) for the sequence \( X_n, Y_n \) in question, and the validity of the equation

\[
\lim_{n \to \infty} \gamma(X_n, Y_n) = \alpha.
\]

Theorem 13 is proved.

The coincidence of the upper and strong lower angles with the angle in the usual sense makes it possible in appropriate situations to apply Theorems 5 and 6 to the ordinary angle.

We note one further consequence of the density of shortest arcs (Theorem 7) and the existence of the angle in the strong sense.

**Theorem 14.** Let \( z(x) \) be the distance from the fixed point \( O \) to the point \( X(x) \) on the simple rectifiable curve \( L \). The parameter \( x \) is the length along \( L \). Suppose, moreover, that at the point \( X(x_0) \) the curve \( L \) has a definite direction to the right. Then at \( x = x_0 \) the right derivative exists and

\[
\left( \frac{\partial z}{\partial x} \right)_r = -\cos \alpha,
\]

where \( \alpha \) is the angle formed by the right branch of \( L \) and by that shortest
arc \( OX(x_0) \) which is the limit of shortest arcs \( OX(x) \) as \( x \to x_0 + 0 \).

This is easily established if the branch of \( L \) is enclosed in an arbitrarily narrow sector between shortest arcs and the results of subsections 4 and 6 of Chapter II are employed.

15. **Preparatory estimates of the difference** \( 2\pi - \theta \). We again return to the consideration of the difference \( 2\pi - \theta \). This time we shall establish the estimates needed in § 7 of this difference in terms of the characteristics of the converging polyhedral metrics.

**Lemma 18.** For every point \( O \) with complete angle \( \theta \) we have the estimate

\[
\theta - 2\pi \leq \liminf_{n \to \infty} \omega_n(U),
\]

where \( U \) is an arbitrarily small neighborhood of the point \( O \) and the \( \omega_n \) are the negative parts of the curvatures of the converging polyhedral metrics.

The finiteness of the lower limit on the right follows from Theorem 10.

In the case \( \theta \leq 2\pi \) the assertion of the lemma is trivial. Suppose that \( \theta > 2\pi \). In accordance with Theorem 8 we decompose a neighborhood of the point \( O \) into angles \( \alpha_i, \pi < \alpha_i < \theta/2 \) by shortest arcs which are limits for the system of shortest arcs serving in the metrics \( \rho_n \) as sides of reduced triangles \( OA_iB_i \) \( (i = 1, \ldots, m) \) surrounding the point \( O \).

From Lemma 15 we will have \( \sum_{i=1}^{m} \alpha_i = \theta \). Each \( \alpha_i \) may be replaced by some \( \gamma_i \), up to \( \varepsilon/m \). The \( \gamma_i \) in turn, for sufficiently large \( n \), may be replaced by \( \gamma_i^n \) with accuracy \( \varepsilon/m \). The quantity \( \gamma_i^n \) does not decrease on being replaced by \( \gamma_i^n_T \). (We do not think it necessary to define the notation \( \gamma_i^n, \gamma_T^n, \gamma_i^n_T, \) which has occurred several times already.)

The quantity \( \gamma_i^n_T \), from Theorem 3, if it decreases at all, decreases by no more than \( \bar{\omega}_n(OA_iB_i) \) on replacement by \( \alpha_i^n \).

The quantity \( \alpha_i^n \) can only increase on replacement by \( \bar{\alpha}_i^n \). Finally, \( \sum \bar{\alpha}_i^n \) cannot exceed \( 2\pi \) by more than the negative part of the curvature at the point \( O \) in the polyhedral metric \( \rho_n \). Thus we have successively found:

\[
\theta = \sum \alpha_i,
\]

\[
\alpha_i \leq \gamma_i + \varepsilon/m,
\]

\[
\gamma_i \leq \gamma_i^n + m/\varepsilon,
\]

\[
\gamma_i^n \leq \gamma_i^n_T,
\]

\[
\gamma_i^n_T - \alpha_i^n \leq \bar{\omega}_n(OA_iB_i),
\]
\[ \alpha_i^a \leq \bar{\alpha}_i^a, \]
\[ \sum \bar{\alpha}_i^a - 2\pi \leq \omega_n^-(O). \]

Adding these inequalities, we obtain
\[ \theta - 2\pi \leq \bar{\omega}_n^+(OA_iB_i) + \omega_n^-(O) + 2\epsilon \leq \omega_n^-(U) + 2\epsilon. \]

By increasing \( n \) we can make the quantity \( \epsilon > 0 \) arbitrarily small and we thus arrive at inequality (18).

**Lemma 19.** For each point \( O \) with complete angle \( \theta \)
\[ 2\pi - \theta \leq \lim \inf_{n \to \infty} \omega_n^+(U), \]
where \( U \) is an arbitrarily small neighborhood of the point \( O \).

For the proof we consider separately the cases \( \theta > 0 \) and \( \theta = 0 \).

Suppose \( \theta > 0 \). In this case the proof is carried out in just the same way as that of the preceding lemma, with some alterations of the corresponding inequalities. The first of these in this case take the form
\[ \gamma_i^a \leq \alpha_i + \frac{\epsilon}{m}, \]
\[ \gamma_i^n \leq \gamma_i^a + \frac{\epsilon}{m}, \]
\[ \gamma_{it}^n = \gamma_i^n. \]

The last equation holds in view of the fact that the sectors at the vertex \( O \) in the triangles \( OA_iB_i \) are so to speak "convex away from \( O \)" (see the proof of Lemma 16 in subsection 11).

Further, from the estimate (3) we have from Theorem 1 that
\[ \bar{\alpha}_i^a - \gamma_{it}^n \leq \omega_n^+(OA_iB_i) \]
and finally
\[ 2\pi - \sum \bar{\alpha}_i^a \leq \omega_n^+(O). \]

Adding all these inequalities, carrying \( \sum \alpha_i = \theta \) to the left and considering that all the triangles \( OA_iB_i \) can be regarded as situated in the neighborhood \( U \), and that \( \epsilon \) may be chosen arbitrarily small, we obtain
\[ 2\pi - \theta \leq \lim \inf_{n \to \infty} \omega_n^+(U). \]

Now suppose \( \theta = 0 \). As in the proof of Lemma 17, we may surround the point \( O \) in a very small neighborhood by a two-gon \( D \), which is limiting for two-gons \( D_n \) surrounding the point \( O \) in the polyhedral metrics \( \rho_n \). In \( D_n \) we join the vertices \( X_n, Y_n \) by shortest arcs to the point \( O \), and the
vertices \( X \) and \( Y \) of the two-gon \( D \) by the limiting shortest arcs \( OX, OY \). The angle between \( OX \) and \( OY \) is equal to zero, since \( \theta = 0 \). The angle \( \gamma(X, Y) \) is then very small (from Theorem 11). Because of the uniform convergence of the metrics, the angles \( \gamma_n(X_n, Y_n) \) are also small for large \( n \). But from Theorem 2 the angles \( \alpha_n^1, \alpha_n^2 \) at the vertex \( O \) cannot exceed \( \gamma_n(X_n, Y_n) \) in any of the triangles \( OX_nY_n \) by more than the positive curvature of these triangles. Therefore we have

\[
\alpha_n^1 + \alpha_n^2 = \theta_n \leq \omega^+_{n}(U - O) + \varepsilon.
\]

Moreover,

\[
2\pi - \theta_n \leq \omega^+_{n}(O).
\]

Adding these inequalities and using the smallness of \( \varepsilon \) for large \( n \), we obtain

\[
2\pi \leq \liminf_{n \to \infty} \omega^+_{n}(U).
\]

which coincides with (19) for \( \theta = 0 \).

Lemma 19 may be strengthened as follows.

**Lemma 20.** If the neighborhood of the point \( O \) is subdivided into convex sectors by successive shortest arcs, forming angles \( \alpha_i \), then the sum of these angles, which in general may be less than \( \theta \), satisfies the inequality

\[
2\pi - \sum \alpha_i \leq \liminf_{n \to \infty} \omega^+_{n}(U),
\]

where \( U \) is any small neighborhood of the point \( O \).

**Proof.** If in at least one of the sectors the shortest arc \( XY \), lying in the sector and joining the points \( X, Y \) on its sides, passes through \( O \), then the corresponding angle \( \alpha_i = \pi \), and the sum of the remaining angles \( \sum_{j \neq i} \alpha_{ij} \leq \pi \). In this case \( \sum \alpha_i \geq 2\pi \) and estimate (20) is trivial.

But if the shortest arcs \( XY \) lie in each sector and do not touch the point \( O \), then for each of the sectors \( \alpha_i = \tilde{\alpha}_i \). In this case \( \sum \alpha_i = \sum \tilde{\alpha}_i = \theta \) and equation (20) follows from (19). Lemma 20 is proved.

**7. Excesses of nonoverlapping triangles.** The fundamental problem of this section is the proof of the following theorem.

**Theorem 15.** Suppose that in a two-dimensional manifold with intrinsic metric \( \rho \), the latter can in the neighborhood of each point be considered as the limit of a uniformly converging sequence of polyhedral metrics, whose positive curvatures are uniformly bounded. Suppose moreover that all the metrics \( \rho_n \) convert \( R \) into one and the same topological space. Then
for each region $G$ with compact closure the sum of the excesses of any finite system of nonoverlapping reduced triangles lying in $G$ is bounded in absolute value by some number depending only on the choice of the region $G$. This is valid both for the excesses with respect to the angles between the sides of the triangles and for the excesses with respect to the angles of the interior sectors of these triangles.

From the assertion of this theorem it follows in particular that in the metric $\rho$ the condition of boundedness of the curvature (subsection 6 of Chapter I) is satisfied. Moreover, it is satisfied in the following strengthened form:

1) not only locally (in some neighborhood of each point) but also in any compact region;

2) not only for simple (homeomorphic to the disc and convex relative to the boundary) triangles, but also for arbitrary reduced triangles;

3) not only the positive excesses, but also the negative excesses of the triangles are bounded;

4) the positive excesses, computed not only with respect to the angles of the triangles, but also with respect to the angles of their interior sectors, are bounded.

We divide up the proof of Theorem 15 into a series of steps. First we consider only the positive excesses of triangles homeomorphic to the disc. Here we are thinking of the excesses computed with respect to the sector angles. Then we consider the analogous negative excesses. Finally, we turn to other forms of reduced triangles and to negative excesses with respect to the angles between the sides of the triangles.

16. Positive excesses of triangles homeomorphic to the disc.

**Lemma 21.** There exists for each point a neighborhood $G$, within which for any system of nonoverlapping triangles homeomorphic to the disc the sum of the positive excesses of these triangles (computed with respect to the angles of the interior sectors) is bounded by some number not depending on the choice of the system of triangles.

**Proof.** Suppose that $G$ is a neighborhood of the point homeomorphic to the disc and that within $G$ the metric $\rho$ is the limit of a uniformly converging sequence of polyhedral metrics $\rho_n$ with bounded positive curvatures: $\omega_n^+(G) \leq C$. Suppose moreover that $\{T_i\}$ is any finite system of nonoverlapping triangles homeomorphic to the disc and lying in $G$. 
1. **Passage to triangles consisting of segments of limiting shortest arcs.**

The sides of the triangles $T$ may not be the limits of shortest arcs joining the same points in the metrics $\rho_n$. This makes it difficult to use the condition of the approximability of $\rho$ by the metrics $\rho_n$. Therefore we shall alter the triangles $T$ somewhat.

On the sides of any triangle $T$ there may be points $X_i$ which are vertices of other triangles $T$ (Figure 63). On the boundary of each triangle we choose points $A_i$, including all the vertices of $T$, the points $X_i$, and a sufficiently dense additional system of points. We successively join the points $A_i$ by shortest arcs in the metrics $\rho_n$ and choose a subsequence of the metrics for which these shortest arcs converge to some limiting shortest arcs. We thus obtain the following situation: 1) the new shortest arcs, successively joining in the metric $\rho$ the points $A_i$ lying on the boundary of $T$, form a triangle $P$ replacing $T$ and having the same vertices; 2) the triangles $P$ also do not overlap; 3) their sector angles are very close to the sector angles of $T$; 4) for the selected subsequence $\rho_n$, the shortest arcs joining the same points $A_i$ in the metrics $\rho_n$ form polygons $P_n$ corresponding to $P$.

In order to guarantee that assertions 1)–4) are satisfied, it suffices to choose the points $A_i$ and the subsequence $\rho_n$ in the following way. First
we mark off the points $X_i$, including the proper vertices of all the triangles $T$. Close to each of the points $X_i$, on all the sides of the triangles issuing from $X_i$, we mark off close points $A'_i$. We join the point $X_i$ to the points $A'_i$ surrounding it in the metrics $\rho_n$ by shortest arcs which have no superfluous intersections, and we choose a subsequence $\rho_n$ for which all of these shortest arcs converge to limiting shortest arcs in $\rho$. This whole construction for each point $X_i$ may be carried out in an arbitrarily small neighborhood of $X_i$, so as not to affect the sides of the triangles $T$ other than those issuing from $X_i$. Moreover, we may because of the closeness of all the $A'_i$ to the $X_i$ guarantee that the sector angles between the newly drawn shortest arcs differ from the corresponding angles of the sectors by arbitrarily small amounts.

Indeed, if the complete angle around a point $X_i$ is equal to zero, then both the original and new sectors at the point $X_i$ have a zero angle and therefore are equal to each other. But if the angle $\theta$ at the point $X_i$ is distinct from zero, then for $A'_i$ close to $X_i$, the old and new shortest arcs $A'_iX_i$, from Theorem 11, form a very small angle, so that the sector they form is also very small. Hence the angle of the sector between the new shortest arcs differs only by little from the angle of the sector between the original shortest arcs.

After the choice of the points $A'_i$, on each of the pieces of the side $T$ we choose additional points $A''_i$ sufficiently often that on successively joining them by shortest arcs we do not encounter curves drawn earlier other than the contiguous segments $A'_iX_i$. The points $A''_i$ are joined successively in the metrics $\rho_n$ by shortest arcs which do not have superfluous intersections with one another and with the shortest arcs drawn earlier. We then select a further subsequence $\rho_n$ for which these shortest arcs converge to some limiting shortest arc in the metric $\rho$.

The system of points $X_i, A'_i, A''_i$ forms in its totality the required system of points $A_i$, and the subsequence of $\rho_n$, beginning with a sufficiently large index, forms the required sequence of $\rho_n$.

The segments of the limiting shortest arcs, together replacing one of the sides of an original triangle $T$, have the same length as the side of $T$. Therefore they also constitute a single shortest arc and therefore in particular form a simple curve. The triangle $T$ is thus replaced by a triangle $P$. This last may now turn out not to be homeomorphic to the disc. Close to the vertices its contiguous sides may coincide, as depicted in Figure 64, i.e., $P$ might be only a reduced triangle.
The polygons \( P_n \) corresponding to the triangle \( P \) in the metrics \( \rho_n \) may of course not be triangles. But they are nonoverlapping polygons. Close to each of their vertices adjacent sides may generally speaking coincide on some sections, depicted in Figure 65.

Because of the arbitrary nearness of the sector angles in the triangles \( T \) and \( P \), it remains for us to prove the boundedness of the excesses for the system of triangles \( P \).

2. Limiting triangulation. The region \( G \) in question has a compact closure \( \overline{G} \). Each point \( X \in \overline{G} \) may by Lemma 16 be enclosed in all the metrics \( \rho_n \) by arbitrarily small absolutely convex polygons \( Q_n \), where the number of their sides for each fixed point \( X \) is bounded by one and the same number independently of the index \( n \), and each polygon \( Q_n \) covers not only the point \( X \), but also some neighborhood \( U \) of \( X \).

From the neighborhoods \( U \) we may select a finite covering of \( \overline{G} \). The corresponding system of polygons \( Q_n \) will cover \( \overline{G} \) and \textit{a fortiori} \( G \), for each \( n \). We may take as included in this collection arbitrarily small polygons \( \hat{Q}_n \), isolatedly covering the neighborhoods of vertices of the polygons \( P_n \), as depicted in Figure 66.

In each of the metrics \( \rho_n \) the system of polygons \( Q_n \) may be broken into a finite number of nonoverlapping polygons \( Q'_n \) which are convex relative to the boundary, in a fashion similar to that of subsection 7 of Chapter III. Here we may suppose that the polygons \( \hat{Q}'_n \) enter unchanged into the system of polygons \( Q'_n \). In the process of the indicated decomposition one may not create superfluous intersections with the
earlier drawn shortest arcs, including the sides of the polygons \( P \).

Now we include the contours of the polygons \( P_n \) in the system of curves of the subdivision. Close to a vertex of \( P_n \) the sides of this polygon separate off pieces \( Q_n' \), which generally speaking are not convex polygons, from the corresponding polygons \( \tilde{Q}_n \). These pieces are crosshatched in Figure 66. The same polygons \( Q_n' \) as are dissected by the various sides of \( P_n \) turn out to be decomposed into convex (relative to the boundary) polygons \( Q_n'' \). Thus each polygon \( P_n \) turns out to be decomposed into nonintersecting polygons \( Q_n'' \), which, with the exception possibly of the polygons \( \tilde{Q}_n'' \), are convex relative to the boundary.

We decompose each polygon \( \tilde{Q}_n'' \) into narrow triangles by shortest arcs issuing from the vertex \( P_n \). We decompose the remaining polygons \( Q_n'' \) by diagonals into triangles convex relative to the boundary. We obtain a triangulation of the polygons \( P_n \). All the triangles of this triangulation with the exception of those adjacent to the vertices \( P_n \) are \textit{a fortiori} convex.

In this way we may triangulate the \( P_n \) in each of the metrics \( \rho_n \). Since the number of initial polygons \( P_n, Q_n \) and their sides is uniformly bounded for all \( n \), then, as one may verify by considering the above description of the triangulation process, the number of elements of the triangulation (vertices, triangles, segments of adjacent sides) will be uniformly bounded for all \( n \).

Because of the uniform boundedness of the number of elements of the triangulation there exists a sequence of metrics \( \rho_n \) for which the triangulations of all the polygons \( P_n \) have the same topological structure, i.e., have the same number of triangles in the triangulation and the same rule for them to be adjacent.

From the above sequence of metrics \( \rho_n \) one may select a still narrower subsequence for which the corresponding vertices and edges converge to certain points and shortest arcs joining them in the metric \( \rho \). The limiting net of edges forms some triangulation of the triangles \( P \) in the metric \( \rho \).

We note that the limiting triangulation does not necessarily have the same structure as the triangulations converging to it. Certain groups of vertices of the triangulations \( P_n \) might converge to one point. This will be one vertex of the limiting triangulation of \( P \). The limiting triangulation consists in this case of a smaller number of elements than the converging triangulations. However, the sides of each triangle of the limiting trian-
gulations will be limits of sides of some definite triangle of the converging triangulations. Therefore, in particular, all the sectors of the triangles of the limiting triangulation adjacent to vertices not lying on the boundary of the triangles $P$ are convex sectors.

Finally, we may suppose that the polygons $Q_n$ and their pieces $Q'_n$ (decomposed into triangles) have been chosen so small that in the limiting triangulation of the triangle $P$ the angles of the triangles of the triangulation at a vertex $P$ are equal to the angles of the sectors of these triangles and in their sum constitute the angles of the interior sector of the triangle $P$.

3. Relations following from Euler’s theorem. Consider one of the triangles $P$ and its limiting triangulation. We denote by $\alpha$ the angles of the triangles $t$ of the triangulation adjacent to the interior vertices of the triangulation and by $\beta$ the angles adjacent to points on the sides of $P$, and by $\xi$ angles adjacent to the vertices of $P$. From Euler’s theorem, as in subsection 9 of Chapter III, we obtain for the sum of the excesses $\delta(t)$ of the triangles of the triangulation the following equation:

$$\sum \delta(t) = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) + \sum (\sum \xi - \pi).$$

Here the sum on the left is extended over all triangles $t$ constituting the subdivision of the triangles $P$. Of the sums on the right, the first is extended over all interior vertices of the triangulation, and the sum $\sum \alpha$ appearing in it relates to the angles between the sides of the triangles encircling that vertex. The second sum is extended over all the vertices of the triangulation lying on the sides of $P$, and the sum $\sum \beta$ over the angles, between the sides of the triangles $t$, adjacent to the corresponding vertex. Finally $\sum \xi$ is the sum of the angles between the sides of triangles $t$ adjacent to the three vertices of $P$. Rewriting (21) in the form

$$\sum \xi - \pi = \sum (2\pi - \sum \alpha) + \sum (\pi - \sum \beta) + \sum \delta(t),$$

we note that the quantity $\sum \xi - \pi$ is simply the excess $\tilde{\delta}(P)$ of the triangle $P$, computed at the angles of its sectors. Therefore the sum of the positive excesses of all the triangles of $P$ is equal to the sum of the positive terms of the right side of equations of the type (22):

$$\sum \tilde{\delta}(P) = \sum (2\pi - \sum \alpha) + \sum (\pi - \sum \beta) + \sum \delta(t).$$

Here the sums are extended over all the triangles $P$ for which the excesses $\tilde{\delta}(P)$ are positive.

Because of the convexity of the sectors adjacent to the interior vertices of the triangulation, we find from Lemma 20 that
(24) \[ \sum (2\pi - \sum \alpha) \leq C, \]
where \( C \) is a common upper bound for the positive parts of the curvature of the approximating metrics \( \rho_n \). The terms \( \pi - \sum \beta \) are nonpositive, since the sum of the angles on one side of a shortest arc cannot be less than \( \pi \):
\[ \sum (\pi - \sum \beta) \leq 0. \]

Thus it remains for us to estimate the excesses \( \delta(t) \) of the triangles of the triangulation.

Each angle \( \alpha \) of any of the triangles \( t \) may with arbitrary accuracy be replaced by the angle \( \gamma \) for some pair of points \( X, Y \) on the sides of that angle. In their turn the angles \( \gamma \) may be approximated, because of the uniform convergence \( \rho_n \to \rho \), by the quantities \( \gamma_n \) for the corresponding points \( X_n, Y_n \) on the sides of that triangle \( t_n \) of the triangulation \( P_n \) which converges to \( t \). Therefore
\[ \sum \delta(t) = \sum (\alpha' + \alpha'' + \alpha''' - \pi) \leq \sum (\gamma' + \gamma'' + \gamma''' - \pi) + \varepsilon \leq \sum (\gamma'_n + \gamma''_n + \gamma'''_n - \pi) + 2\varepsilon. \]

Each of the angles \( \gamma_n \) does not decrease if it is measured inside the triangle \( t_n \), so that
\[ \sum \delta(t) \leq \sum (\gamma'_n + \gamma''_n + \gamma'''_n - \pi) + 2\varepsilon. \]

Finally, each angle \( \gamma_n \) decreases by no more than \( \omega_n(t_n) \) on swinging the triangle \( t_n \) onto the plane. But after doing this the excess of this triangle is equal to zero, so that
\[ \sum \delta(t) \leq 3 \sum \omega_n(t_n) + 2\varepsilon, \]
and in view of the arbitrariness of \( \varepsilon \) and the estimate \( \sum \omega_n(t_n) \leq C \) we obtain
\[ \sum \delta(t) \leq 3C. \]

From equation (23) and inequalities (24), (25), and (27), it follows that
\[ \sum \delta(t) \leq 4C, \]
which proves Lemma 21.

Lemma 21 has a number of important consequences.

**Corollary 1.** The manifold in question is a manifold of bounded curvature in the sense of the definition of subsection 6 of Chapter I.

**Corollary 2.** By the theorem of subsection 17 of Chapter III we may consider the limit, not only locally, but in any region \( G \) with a compact
closure, of uniformly converging polyhedral metrics \((p_n)_P\), defined in a
polygon \(P\) containing \(G\) and having absolute curvatures bounded uniformly,
while close to the interior points of the region \(G\) these metrics will converge
to the metric \(p\).

**Corollary 3.** Repeating the proof of Lemma 21 for a compact region
\(G\) and the polyhedral metrics \((p_n)_P\) indicated above, we may verify that not
only locally, but also for any compact region \(G\) and any system of non-
overlapping triangles \(T_i \subseteq G\) homeomorphic to discs we have
\[
\sum_i \delta(T_i) \leq 4C(P).
\]

17. Negative excesses of triangles homeomorphic to the disc.

**Lemma 22.** For any finite system of nonoverlapping triangles homeomor-
phic to the disc, the sum of the negative excesses, computed at the angles
of the interior sectors of these triangles, is bounded by one and the same
number, depending only on the choice of the region \(G\):
\[
\sum \delta^-(T_i) \leq D,
\]
where \(D\) depends on the common estimate of the negative parts of the
curvature of the polyhedral metrics by which one may uniformly approximate
the metric \(p\) in a polygon \(G'\) enclosing \(G\).

For the proof, as in Lemma 21, we turn from the triangles \(T\) to
somewhat altered triangles \(P\) and polygons \(P_n\) converging to them. Then
we construct triangulations of \(P_n\) of the same type and a limiting trian-
gulation of the triangles \(P\). Then we write down relation (21) for each
of them, this time in the form
\[
\pi - \sum \xi = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) - \sum \delta(t).
\]

Summing this relation over all triangles in \(P\) having negative excesses,
we obtain
\[
\sum \delta^-(P) = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) - \sum \delta(t),
\]
where the sums on the right are extended over the triangulations of all
triangles \(P\) with negative excesses.

But for the first of the sums we have from Lemma 18:
\[
\sum (\sum \alpha - 2\pi) \leq \sum (\theta - 2\pi) \leq \lim inf_{n \to \infty} \omega_n(U),
\]
where the \(U\) are arbitrarily small neighborhoods of the interior vertices
of the triangulation.

For the second sum we have:
\[
\sum (\sum \beta - \pi) \leq \sum (\sum \bar{\beta} - \pi) \leq \lim inf_{n \to \infty} \omega_n^-(U),
\]
where the $U$ are arbitrarily small neighborhoods of those vertices of the triangulation which lie on the sides of $P$.

It remains to estimate the sum of the negative excesses over the angles between the sides of the triangles of the triangulation themselves.

As in the preceding lemma, the angles $\alpha', \alpha'', \alpha'''$ of each triangle $t$ may be approximately replaced by the angles $\gamma', \gamma'', \gamma'''$ for any pair of points sufficiently close to the vertices on the sides of these triangles. These angles may be approximately replaced by angles $\gamma''', \gamma'', \gamma'$ for the corresponding points on the sides of the corresponding triangles $t_n$ in the metrics $\rho_n$. The basic difficulty remains, namely that of passing from the angles $\gamma''', \gamma'', \gamma'$ to the angles $\gamma''', \gamma'', \gamma'$ obtained by measuring the distance between the same points inside the triangle $t_n$ itself.

It would seem that the angles may increase on this substitution, which would not suit us, since we would then be faced with obtaining an upper estimate for the expressions $\pi - (\gamma''' + \gamma'' + \gamma')$ from the corresponding estimate for $\pi - (\gamma''' + \gamma'' + \gamma')$.

So let us consider the detailed transition from $\gamma_n$ to $\gamma_t$ for various angles of the triangles $t_n$.

1. Close to interior vertices of the triangulations of the polygons $P_n$ the sectors of the triangles $t_n$ are convex. Therefore for such angles $\gamma_n = \gamma_t$.

2. If the vertex $O$ of the corresponding triangle $t$ lies on the side of $P$, then the shortest arcs, for fixed (as to distance from the vertex) points $X_n, Y_n$ on the sides of $t_n$ for large $n$ cannot envelop the vertex $O$, since otherwise we would obtain in the limit a shortest arc $X, Y$ enveloping $O$. But then there exists a shortest arc $\overline{XY}$ passing through $O$, and already that angle alone at the vertex $O$ in the triangle $t$ is equal to $\pi$. But we have been considering only triangles $t$ with negative excesses. Thus we may also assume that close to such vertices $\gamma_n = \gamma_t$.

3. If we are dealing with an angle of a triangle adjacent to a vertex of $P$ in which the complete angle $\theta > 0$, then, since the angle of the triangle $t$ was chosen less than $\pi$ and less than $\theta/2$, the sectors of the triangles $t_n$, as already noted in the proof of Lemma 16, are so to speak convex at some distance from the vertex. Therefore in this case as well we may suppose that $\gamma_n = \gamma_t$.

4. Finally, if the vertex $O$ of the triangle $t$ coincides with a vertex of $P$ and at that vertex $\theta = 0$, then on the basis of Lemma 17 we may encircle the point $O$ in the metrics $\rho_n$ with loops of two shortest arcs, converging to an analogous loop in the metric $\rho$. All of these loops lie close to $O$
and at the same time are arbitrarily many times smaller in perimeter than in distance to the point \( O \). We choose points \( X, Y \), limiting for points \( X_n, Y_n \) lying on the intersection of these loops with the sides of the triangle \( t_n \). Then we verify that the angles \( \gamma_n \) are insignificantly small and therefore they cannot exceed the \( \gamma_n \) by any more than an arbitrary \( \varepsilon > 0 \) given in advance.

Therefore for such vertices we may suppose that

\[
\gamma_i - \gamma_n \leq \varepsilon.
\]

The number of such vertices is finite. Incidentally, it is easy to verify that we may assume \( \gamma_i > \gamma_n \) in general for not more than one of the triangles \( t_n \) adjacent to a vertex of \( P \).

All of this shows that for the estimation of \( \sum \hat{\delta}^- (t_i) \) it is sufficient to estimate \( \sum (\pi - \gamma_i' - \gamma_i'' - \gamma_i''') \). But for these quantities we have the right to use the results of Theorems 1 and 3, which, as in Lemma 21, leads us to the required estimate.

18. Excesses of arbitrary reduced triangles.

Lemma 23. The assertions of Lemmas 21 and 22 are valid not only for reduced triangles homeomorphic to the disc, but also for arbitrary non-overlapping triangles.

1. For a triangle degenerating into a segment the angle of a sector on the side of the triangle for a vertex lying inside the segment is not defined. In this case both adjacent sectors are “exterior” for the triangle. For such triangles we consider the excess to be by definition equal to zero.

2. Consider positive excesses for other types of triangles. Exterior tails of triangles may be simply dropped. This does not increase the angles of the sectors on the side of the triangles. For the remaining triangles, homeomorphic to the disc or having interior tails, the same considerations as in the proof of Lemma 21 apply.

3. Consider negative excesses with respect to the angles of sectors for triangles of various types. If the triangle has exterior tails, then the rejection of these tails may somewhat increase the sum of the angles of the interior sectors of that triangle, by an amount equal to the angle of the interior sector \( \alpha \) at the newly appearing vertex \( D \) (see Figure 67). This last quantity is the contribution of this sector to the excess \( \theta - 2\pi \) at the point \( D \). But such an excess may be estimated, using Lemma 18, in terms of the quantities \( \omega_n(U) \) for small neighborhoods \( U \) of the point.
D. These curvatures play no essential role in the other estimates of Lemma 22.

Figure 67.

If the triangle consists only of three tails (Figure 67), it has a negative excess equal to \(-\pi\). But no matter how many such nonoverlapping triangles are adjacent to the point \(O'\), each of them gives a contribution not less than \(\pi\) in the difference \(\theta - 2\pi\) at the point \(O'\). Therefore, also for such triangles, we obtain an estimate of the quantity \(\delta^-(T)\) in terms of the quantity \(\omega_n(U(O'))\).

To the remaining triangles homeomorphic to the disc and triangles with interior tails one may apply the same considerations as in Lemma 22.

Lemma 23 is thus proved.

Lemma 24. In a compact region \(G\), for any finite system of nonoverlapping reduced triangles \(T_i \subset G\), not only the sum of the negative excesses with respect to the sector angles of these triangles, but also the sum of the negative excesses over the angles between the sides of these triangles, are uniformly bounded in absolute value.

Proof. In the cases when the angle \(\alpha\) between the sides is equal to the angle \(\tilde{\alpha}\) of the interior sector of the triangle, we replace \(\alpha\) by \(\tilde{\alpha}\). When at least one of the angles of the triangle \(\alpha \geq \pi\), the excess of the triangle is nonnegative so that in general this triangle need not be taken into consideration.

There remain angles for which \(\alpha < \pi, \alpha < \tilde{\alpha}\). But always

\[
\alpha = \min\{\pi, \tilde{\alpha}, \tilde{\beta}\},
\]

In Lemma 22 there figured quantities \(\omega_n(U)\) for neighborhoods of interior vertices of the triangulation. But the point \(O\) cannot be such a vertex.
where \( \bar{\beta} \) is the angle of the exterior sector, so that in the indicated cases \( \alpha = \bar{\beta} \). Then we have:

\[
\bar{\alpha} - \alpha = \bar{\alpha} - \bar{\beta} = 2(\bar{\alpha} - \pi) + (2\pi - \bar{\alpha} - \beta) = 2(\bar{\alpha} - \pi) + (2\pi - \theta),
\]

where \( \theta \) is the complete angle around the vertex \( O \) of the angle \( \alpha \).

But by Lemmas 21 and 19

\[
\sum (\alpha - \pi) \leq \sum \delta(T_i) \leq 4C(P),
\]

\[
\sum (2\pi - \theta) \leq \sum \lim_{n \to \infty} \omega^+(U(O)) \leq C(P),
\]

so that

\[
\sum (\bar{\alpha} - \alpha) \leq 9C(P).
\]

Thus we finally obtain

\[
\sum \delta^-(T_i) \leq \sum \delta^-(T_i) + 9C(P) \leq D + 9C(P),
\]

and Lemma 24 is proved.

The fundamental Theorem 15 follows from Lemmas 21–24.
1. **On a method of introducing measure.**

1. *Definition of measure with the aid of a function given on nonoverlapping sets.* Suppose in a metric space \( R \) that we are given a system \( S \) of sets \( t \), each of which is a continuum, i.e., is a connected closed compact set in the space \( R \). We suppose that the empty set \( 0 \in S \). As \( t \) we may consider subsets of \( R \) supplemented by certain rules, for example a closed segment with a distinguished interior point. In this case different sets \( t \) may, while coinciding as sets, differ in their supplementary rules.

Suppose that for certain pairs \( t_i, t_j \in S \) the relation or “nonoverlapping,” of “the absence of essential intersections,” is defined, which we will denote by the notation \( t_i \not\subset t_i \), and suppose that this relation satisfies the axiom:

1) if \( t_i \not\subset t_i \), then \( t_j \not\subset t_i \);
2) if \( t_i \cap t_j = 0 \) then \( t_i \not\subset t_j \);
3) for each \( t \in S \) there is an arbitrarily fine subdivision into nonoverlapping \( t_i \in S \), with the property that \( t_i \not\subset t' \in S \) if \( t \not\subset t' \). A subdivision satisfying the last requirement is said to be regular.

**Example 1.** \( R \) is a straight line and the \( t \) are its closed segments, and nonoverlapping means the absence of common interior points.

**Example 2.** \( R \) is a two-dimensional manifold of bounded curvature, \( t \) are reduced triangles, and nonoverlapping is understood in the sense of subsection 2 of Chapter III. In Figure 68 we may observe the difference between regular and nonregular decompositions. The reduced triangles \( ABC \) and \( ADE \) are nonoverlapping. But \( LMN \) and \( ABC \) are overlapping. The decomposition of \( ADE \) into triangles, including \( LMN \), is not regular.

**Example 3.** \( R \) is a two-dimensional manifold of bounded curvature, and the \( t \) are reduced triangles and points; nonoverlapping for points means that they do not coincide, nonoverlapping
for a point and a triangle means that the point is situated outside the triangle or at a vertex of it, and nonoverlapping of triangles is understood as in Example 2.

We proceed with the general discussion. Sets representable in the form of a sum of a finite number of pairwise nonoverlapping \( t_i \in S \) are denoted by \( P \), and the system of such sets by \( \{P\} \). One and the same \( P \in \{P\} \) may have various representations in the form of finite sums of nonoverlapping \( t_i \). A separate such decomposition is denoted by \( T_P \). The largest of the diameters of sets \( t_i \in T_P \) is denoted by \( d(T_P) \).

Now we suppose that a nonnegative function \( \phi(t_i) \) is defined on the sets \( t_i \). We suppose that \( \phi(0) = 0 \).

We define a function

\[
\mu_0(P) = \lim_{d(T_P) \to 0} \sup \sum_{t_i \in T_P} \phi(t_i)
\]

on the sets \( P \). There may always exist finite or infinite values of \( \mu_0(P) \).

We define further for open sets \( G \) the function

\[
\mu_1(G) = \sup \mu_0(P_0)
\]

and for any set \( M \subset R \) the function

\[
\mu(M) = \inf_{G \supseteq M} \mu_1(G).
\]

**Theorem 1.** If the function \( \phi(t) \) is such that for each \( P \in \{P\} \) and each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that each subdivision \( T_P \) of diameter less than \( \delta \) admits an arbitrarily fine further subdivision \( T'_P \) into nonoverlapping sets \( t_i \in S \) for which

\[
\sum_{t_i \in T'_P} \phi(t_i) \geq \sum_{t_i \in T_P} \phi(t_i) - \varepsilon,
\]

then the function \( \mu(M) \) defined by equation (3) is a Carathéodory measure.

It follows from Theorem 1 that the function \( \phi(M) \) is completely additive on the ring of Borel sets. Moreover, it follows from (3) that \( \mu(M) \) is a regular Carathéodory measure and that for each \( M \) there exists a set \( H \) of type \( G_\delta \) containing \( M \) for which \( \mu(H) = \mu(M) \).

2. Proof of Theorem 1. 1. A finite or infinite value \( \mu(M) \) is defined for each \( M \subset R \), while \( \mu(M) \geq 0 \) and \( \mu(0) = 0 \). Therefore for the proof of Theorem 1 it remains to verify three conditions (see [70], Chapter II).

I. \( \mu(M) \leq \mu(N) \) for \( M \subset N \);

II. \( \mu\left( \bigcup_{i=1}^{\infty} M_i \right) \leq \sum_{i=1}^{\infty} \mu(M_i) \) for any sequence \( M_i \);
III. \( \mu(M_1 \cup M_2) = \mu(M_1) + \mu(M_2) \) if \( \rho(M_1,M_2) > 0 \), where \( \rho \) is the distance in the space \( \mathbb{R} \).

2. From (3) it is obvious that condition I and the equality \( \mu(G) = \mu_1(G) \) hold.

3. Lemma 1. For nonoverlapping open sets \( G_1, G_2 \)

\[ \mu_1(G_1) + \mu_1(G_2) = \mu_1(G_1 \cup G_2). \]

Indeed, for any \( P_1 \subset G_1 \) and \( P_2 \subset G_2 \), their sum \( P_1 \cup P_2 = P \) is contained in \( G_1 \cup G_2 \) and belongs to \( \{P\} \). Evidently also \( \mu_0(P) \geq \mu_0(P_1) + \mu_0(P_2) \). Therefore

\[ \mu_1(G_1) + \mu_1(G_2) \leq \mu_1(G_1 \cup G_2). \]

On the other hand, every \( P \subset G_1 \cup G_2 \) decomposes into parts \( P_1 \subset G_1 \) and \( P_2 \subset G_2 \). Each of these parts consists of entire connected components of the set \( P \). Therefore each piece consists of entire sets \( t_i \) and therefore \( P_1, P_2 \in \{P\} \). Moreover, by an analogous principle (here we are using the connectedness of the sets \( t_i \)),

\[ \mu_0(P) = \mu_0(P_1) + \mu_0(P_2). \]

Therefore

\[ \mu_1(G_1) + \mu_1(G_2) \geq \mu_1(G_1 \cup G_2). \]

Along with the preceding inequality this proves equation (5).

4. Now we shall verify the validity of condition III. Suppose that \( G_1 \) and \( G_2 \) are nonintersecting open sets containing \( M_1 \) and \( M_2 \) respectively. Such sets exist, since \( \rho(M_1,M_2) > 0 \). Suppose, moreover, that \( G'_1 \) and \( G'_2 \) are open sets containing \( M_1 \) and \( M_2 \) for which the values of \( \mu_1 \) exceed respectively \( \mu(M_1) \) and \( \mu(M_2) \) by not more than \( \varepsilon > 0 \). (So far we are considering cases when \( \mu(M_1) \) and \( \mu(M_2) \) are finite.) Then we have:

\[ \mu_1(G'_1 \cup G'_2) \leq \mu_1(G'_1 \cup G'_2) = \mu_1(G'_1 \cup G'_2) + \mu_1(G'_2 \cup G'_2) \]

\[ \leq \mu_1(G'_1) + \mu_1(G'_2) \leq \mu(M_1) + \mu(M_2) + 2\varepsilon. \]

The finiteness of \( \mu(M'_1 \cup M'_2) \) follows from inequality (6), and therefore there exists an open set \( G \) for which the first of the following chain of inequalities is valid:

\[ \mu(M_1 \cup M_2) + \varepsilon \geq \mu_1(G) \geq \mu_1(G'_1 \cup G'_2 \cup G'_2 \cup G'_2) \]

\[ = \mu_1(G'_1 \cup G'_2 \cup G'_2 \cup G'_2) \geq \mu(M_1) + \mu(M_2). \]

From inequalities (6) and (7), because of the arbitrary smallness of \( \varepsilon > 0 \), it follows that condition III is satisfied for the sets \( M_1, M_2 \).

If one of the values \( \mu(M_1), \mu(M_2) \), say the first, is infinite, then...
\[
\mu(M_1 \cup M_2) \leq + \infty = \mu(M_1) + \mu(M_2)
\]

and

\[
\mu(M_1 \cup M_2) \geq \mu(M_1) = + \infty = \mu(M_1) + \mu(M_2).
\]

Therefore condition III is valid in this case too.

5. **Lemma 2.** For any expanding sequence of open sets \(G_1 \subset G_2 \subset G_3 \subset \cdots\), we have the equation

\[
\mu_{\infty} \left( \bigcup_{i=1}^{\infty} G_i \right) = \lim_{i \to \infty} \mu_1(G_i).
\]

**Proof.** Suppose that the quantity on the left is finite. Choose in \(\bigcup_{i=1}^{\infty} G_i\) a set \(P \in \{P\}\) such that \(\mu_0(P) \geq \mu_0(\bigcup_{i=1}^{\infty} G_i) - \varepsilon\). Each point of the set \(P\) belonging to \(\bigcup_{i=1}^{\infty} G_i\) belongs, along with some neighborhood, to the set \(G_i\). From the covering of a compact \(P\) by such neighborhoods we may select a finite covering. Therefore all the \(G_i\) for \(i \geq j\) contain \(P\) and for \(i \geq j\) we have:

\[
\mu_{\infty} \left( \bigcup_{i=1}^{\infty} G_i \right) \geq \mu_1(G_i) \geq \mu_0(P) \geq \mu_{\infty} \left( \bigcup_{i=1}^{\infty} G_i \right) - \varepsilon,
\]

from which (8) follows.

If \(\mu_{\infty}(\bigcup_{i=1}^{\infty} G_i) = + \infty\), then we choose \(P\) so that \(\mu_0(P) \geq N\), where \(N\) is an arbitrarily large number. As above, we verify that, beginning with some \(i\), we have \(\mu_1(G_i) \geq \mu_0(P) \geq N\), from which it follows that the right side of (8) is also infinite.

6. **Lemma 3.** If a compact set \(M\) in a metric space \(R\) is contained in the sum of the open sets \(M \subset \bigcup G_n\), then there exists an \(\varepsilon > 0\) such that each piece \(m \subset M\) with diameter \(d(m) \leq \varepsilon\) is entirely contained in one of the sets \(G_n\).

Suppose the contrary. Then there exists a sequence of sets \(m_i \subset M\) for which \(d(m_i) \to 0\) and no \(m_i\) belongs to any \(G_n\). By the compactness, there exists in \(M\) a condensation point \(A\) for the \(m_i\). Along with \(M\) the point \(A\) belongs to \(\bigcup G_n\) and therefore to one of the sets \(G_n\), moreover along with some neighborhood. But this last contradicts the assumed properties of the \(m_i\).

7. **Lemma 4.** For any open sets

\[
\mu_1(G_1) + \mu_1(G_2) \geq \mu_1(G_1 \cup G_2).
\]
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Proof. We choose in \( G_1 \cup G_2 \) a set \( P \in \{ P \} \) for which \( \mu_0(P) \geq \mu_1(G_1 \cup G_2) - \varepsilon \). (If \( \mu_1(G_1 \cup G_2) = \infty \), we replace \( \mu_1(G_1 \cup G_2) - \varepsilon \) by an arbitrarily large \( N \). This does not change the discussion which follows.)

Suppose that \( \delta > 0 \) is chosen for the set \( P \) and the number \( \varepsilon \) in accordance with condition (4) of Theorem 1 of Chapter V, and suppose that \( T_P \) is a decomposition of \( P \) into nonoverlapping sets \( t_i \) of diameter smaller than \( \delta \), for which \( \sum_{t_i \in T_P} \phi(t_i) \geq \mu_0(P) - \varepsilon \). We may in addition suppose that all the \( t_i \in T_P \) are so small that in accordance with Lemma 3 each of the sets \( t_i \) lies entirely in one of the sets \( G_1, G_2 \). We form two sets \( P_1, P_2 \in \{ P \} \) by relating to \( P_1 \) those \( t_i \in T_P \) which lie in \( G_1 \) and to \( P_2 \) the remaining \( t_i \in T_P \).

Since the above inequality is preserved by condition (4) of Theorem 1, we may suppose that the decomposition \( T_P \) is so fine that

\[
\mu_0(P_1) + \mu_0(P_2) \geq \sum_{t_i \in T_P} \phi(t_i) - \varepsilon.
\]

Therefore we will have

\[
\mu_1(G_1) + \mu_1(G_2) \geq \mu_0(P) + \mu_0(P) \geq \sum_{t_i \in T_P} \phi(t_i) - \varepsilon
\]

\[
\geq \mu_0(P) - 2\varepsilon \geq \mu_1(G_1 \cup G_2) - 3\varepsilon.
\]

Because of the arbitrary smallness of \( \varepsilon > 0 \) it therefore follows that inequality (9) holds.

8. We now show that condition II holds:

(10)

\[
\mu \left( \bigcup_{i=1}^{\infty} M_i \right) \leq \sum_{i=1}^{\infty} \mu(M_i).
\]

If the right side of (10) is infinite, this inequality is trivial. Suppose that all the \( \mu(M_i) \) are finite. We choose a covering \( G_i \supseteq M_i \) such that

\[
\mu_1(G_i) \leq \mu(M_i) + \frac{\varepsilon}{2^i}.
\]

Write \( \bigcup_{k=1}^{i} G_k = G_i' \). Then we have

\[
\mu \left( \bigcup_{i=1}^{\infty} M_i \right) \leq \mu_1 \left( \bigcup_{i=1}^{\infty} G_i \right) = \mu_1 \left( \bigcup_{i=1}^{\infty} G_i' \right)
\]

which by Lemma 2

\[
= \lim_{i \to \infty} \mu_1(G_i')
\]

and for sufficiently large \( n \).
\[ \leq \mu_1(G_n') + \varepsilon = \mu_1\left(\bigcup_{i=1}^{n} G_i\right) + \varepsilon \]

or by Lemma 4

\[ \leq \sum_{i=1}^{n} \mu_1(G_i) + \varepsilon \leq \sum_{i=1}^{\infty} \mu_1(G_i) + \varepsilon \]

and by the choice of \( G_i \)

\[ \leq \sum_{i=1}^{\infty} \mu(M_i) + 2\varepsilon. \]

Because of the arbitrary smallness of \( \varepsilon > 0 \) inequality (10) therefore follows.

Theorem 1 is completely proved.

3. Simplest consequences of Theorem 1.

**Theorem 2.** If for each \( P \in \{P\} \) there exists the limit

\[ \mu'_0(P) = \lim_{d(T_P) \to 0} \sum_{i \in T_P} \phi(t_i), \]

then the function

\[ \mu(M) = \inf_{G \supseteq M} \sup_{P \subseteq G} \mu'_0(P) \]

is a regular Carathéodory measure.

In view of the existence of the limit (11), under the conditions of Theorem 2 inequality (4) of Theorem 1 is satisfied. Moreover, in this case \( \mu'_0(P) = \mu_0(P) \). Therefore Theorem 2 follows immediately from Theorem 1.

**Theorem 3.** If each set \( t \in S \) admits an arbitrarily fine subdivision \( T_P \) into nonoverlapping sets \( t_i \in S \), under which \( \sum_{i \in T_P} \phi(t_i) \geq \phi(t) - \varepsilon \) where \( \varepsilon \) is an arbitrarily small positive number, then the function

\[ \mu(M) = \inf_{G \supseteq M} \mu'_1(G), \]

where

\[ \mu'_1(G) = \sup_{T \subseteq G} \sum_{t_i \in T} \phi(t_i), \]

is a regular Carathéodory measure. \( T \) on the right side of (14) refers to a finite system of nonoverlapping \( t_i \in S \).

**Lemma 5.** Under the conditions of Theorem 3

\[ \mu'_1(G) = \mu_1(G). \]
Indeed, in view of definitions (1) and (2) there is a sequence $P_n \subseteq \{P\}$ in $G$ and subdivisions $T_P$, for which $\sum_{i \in T_n} \phi(t_i) \to \mu_1(G)$. Therefore we always have $\mu'_1(G) \geq \mu_1(G)$.

Moreover, under the conditions of Theorem 3, for each system $T_P \subseteq G$ we have $\mu_0(P) \geq \sum_{i \in T_P} \phi(t_i)$. Therefore $\mu_1(G) \geq \mu'_1(G)$. Along with the preceding inequality this yields (15).

Theorem 3 now follows directly from Theorem 1, since in this case condition (4) of Theorem 1 is immediately verified, and $\mu'_1(G)$ coincides with $\mu_1(G)$.

We denote the function $\mu$ defined by equations (13), (14) with a given system $S = \{t\}$ and a relation of nonoverlapping of sets $t$ defined in it, and with the function $\phi(t)$ given, by $\mu_{S, \phi}$.

**Theorem 4.** Suppose that there are selected in the space $R$ two systems $S$ and $\bar{S}$ of sets as defined at the beginning of this section, and that on the sets which enter there are defined respectively functions $\phi$ and $\bar{\phi}$, with the following conditions satisfied: $S \subset \bar{S}$; for $t \in S$ always $\phi(t) \leq \bar{\phi}(t)$; if $t_i \cap t_j$ in $S$, then $t_i \cap t_j$ in $\bar{S}$. If, moreover, each set $t \in \bar{S}$ for any $\varepsilon > 0$ contains a system of nonoverlapping sets $t_i \in S$ for which

\begin{equation}
\sum \phi(t_i) \geq \bar{\phi}(t) - \varepsilon,
\end{equation}

then

\begin{equation}
\mu_{S, \phi} = \mu_{S, \bar{\phi}}.
\end{equation}

**Proof.** Since the system $\bar{S}$ contains $S$ and $\bar{\phi} \geq \phi$ then from (14) and (15) it follows that $\mu_{S, \phi} \leq \mu_{S, \bar{\phi}}$. The reverse inequality follows from equation (16), so that equation (17) follows.

**Remark.** 1) If under the conditions of Theorem 4 one of the functions $\mu_{S, \phi}$, $\mu_{S, \bar{\phi}}$ is a regular Carathéodory measure, then the second function has the same property.

2) Condition (16) in Theorem 4 may be replaced by another: for each finite system of nonoverlapping $t_i \in \bar{S}$, lying in the open set $G$, and for any $\varepsilon > 0$, there must exist a finite system of nonoverlapping $t_j \in S$ contained in $G$ for which $\sum \phi(t_i) \geq \sum \bar{\phi}(t_i) - \varepsilon$.

2. **Definition of curvature.**

4. **Positive and negative parts of the curvature.** We turn to the consideration of an arbitrary two-dimensional manifold of bounded curvature. Suppose that $S_t$ is a family of reduced triangles $t$ convex relative to the
boundary, and that \( \{t\} \) is a finite selection of nonoverlapping triangles \( t \).

The positive part of the curvature, \( \omega^+(G) \), of an open set \( G \) is the least upper bound of the sums of positive excesses for finite systems of nonoverlapping triangles \( t \in S_1 \) contained in \( G \). Here the excess is defined with respect to the angles of the sectors of the triangles:

\[
\omega^+(G) = \sup_{\{t\} \subset G} \sum_{t_i \in \{t\}} \delta^+(t_i).
\]

The corresponding definition for the negative part of the curvature of the open set is:

\[
\omega^-(G) = \sup_{\{t\} \subset G} \sum_{t_i \in \{t\}} \delta^-(t_i),
\]

where \( \delta^-(t_i) \) is the absolute value of the negative excess, defined with respect to the sector angles, of the triangle \( t_i \), or zero if the excess \( \delta(t_i) \) is positive.

For an arbitrary set \( M \) we put by definition

\[
\omega^+(M) = \inf_{G \supseteq M} \omega^+(G), \quad \omega^-(M) = \inf_{G \supseteq M} \omega^-(G).
\]

Remark. For an open set the last definition does not contradict definitions (18) and (19). These definitions coincide also with the definitions of subsection 7 of Chapter I for the case of a polyhedral metric.

Lemma 6. Every triangle \( t \in S_1 \), for any \( \varepsilon > 0 \), may be subjected to a regular subdivision into arbitrarily fine nonoverlapping \( t_i \in S_1 \) so that the condition

\[
\left| \sum \delta(t_i) - \delta(t) \right| < \varepsilon
\]

holds.

Proof. From subsection 15 of Chapter IV, in the compact set \( t \) there exists not more than a countable collection of points \( A_i \) at which the complete angles \( \theta_i \equiv 2\pi \), while \( \sum_{i} \left| 2\pi - \theta_i \right| < \infty \). We choose a finite number of points \( A_i (i = 1, 2, \cdots, n) \), such that for any choice of other points \( \sum |2\pi - \theta_i| \leq \varepsilon \). Those of the points \( A_i \) which lie inside \( t \) we encircle by polygons \( P_i \). We may suppose that the \( P_i \) are arbitrarily small, lie in \( t \), have no common points, and are convex relative to the boundary (see subsection 11 of Chapter III). For the points \( A_i \) lying on the side of \( t \), as the polygons \( P_i \), we choose analogous closed semineighborhoods adjacent to the side \( t \).

As in the proof in subsection 16 of Chapter III, we may first decompose \( t \) into polygons convex relative to the boundary, among whose number
the $P_i$ all fall, and then decompose each polygon by diagonals issuing from its vertex into reduced triangles $t_i \subseteq S_i$.

An application of Euler's theorem to the decomposition thus obtained, as in subsection 9 of Chapter III, leads to the relation

\[(22) \quad \sum \delta(t_i) = I + II + \delta(t),\]

where the terms $I$ and $II$ have the following sense.

For each interior vertex of the subdivision we define the sum $\sum \bar{\alpha}_i - 2\pi$, where the $\bar{\alpha}_i$ are the sector angles of the triangles of the subdivision adjacent to the vertex, if the latter adjoin it at vertices. For a triangle adjoining the vertex in question at an interior point of its side, the corresponding sector angle is replaced by the angle $\pi$. Expression $I$ is the sum

\[(23) \quad I = \sum (\sum \bar{\alpha}_i - 2\pi),\]

extended over all the interior vertices of the subdivision.

For each vertex of the subdivision lying on a side of $t$, we define the sum $\sum \bar{\beta}_i - \pi$, where the $\bar{\beta}_i$ are the sector angles of the triangles of the subdivision which adjoin the vertex in question of the subdivision. If a triangle in the subdivision adjoins the vertex in question at an interior point of its side, in the sum $\sum \bar{\beta}_i$ we include the angle $\pi$ instead of the corresponding sector. Expression $II$ is the sum

\[(24) \quad II = \sum (\sum \bar{\beta}_i - \pi),\]

extended over all the vertices of the subdivision lying on the sides of $t$.

By the construction all the enumerated vertices of the subdivision do not lie at the points $A_1, \ldots, A_n$, and for them $\sum |2\pi - \theta| < \varepsilon$. This makes it possible to conclude that $|I + II| < \varepsilon$. Therefore, inequality (21) follows from equation (22). Lemma 6 is proved.

From inequality (21) it follows that for the same subdivision

$$\sum \delta^+(t_i) \geq \delta^+(t) - \varepsilon, \quad \sum \delta^-(t_i) \geq \delta^-(t) - \varepsilon.$$ 

In other words, for $S_i$ and $\delta^+$, and also for $S_i$ and $\delta^-$, the conditions of Theorem 3 are satisfied. This leads us to the following assertion.

**Theorem 5.** The functions $\omega^+$ and $\omega^-$ are regular Carathéodory measures.

From Theorem 15 of Chapter IV, these measures take on finite values for every set lying in a region with a compact closure.
5. Curvature and absolute curvature. The curvature $\omega(M)$ of the set $M$ is the difference
\begin{equation}
\omega(M) = \omega^+(M) - \omega^-(M).
\end{equation}
Evidently, $\omega(M)$ is generally speaking a set function of variable sign, defined for every set for which at least one of the functions $\omega^+(M)$, $\omega^-(M)$ is finite. It is completely additive in the following sense. If $\omega^+(M)$ or $\omega^-(M)$ is finite and if $M$ is the sum of nonintersecting sets $M_i$ ($i = 1, 2, \cdots$), then $\omega(M) = \sum_{i=1}^{\infty} \omega(M_i)$.

The absolute curvature $\Omega(M)$ of the set $M$ is the sum
\begin{equation}
\Omega(M) = \omega^+(M) + \omega^-(M).
\end{equation}
Evidently $\Omega(M)$ is a Carathéodory measure and
\[ \Omega(M) = \inf_{\Omega \supset M} \Omega(G). \]

3. Other definitions of curvature.

6. Auxiliary propositions. The proof of Lemma 6 does not change if we consider instead of the triangle $t \in S_1$ an arbitrary reduced geodesic triangle whose sectors at vertices are convex relative to the boundary. If we denote the family of these latter triangles by $S_2$, then $S_1 \subset S_2$.

Moreover, analogously to Lemma 6, for $t \in S_2$ and any $\varepsilon > 0$ there exists a regular subdivision of $t \in S_2$ into arbitrarily fine $t_i \in S_1$ for which
\[ \sum \delta^+(t_i) \geq \delta^+(t) - \varepsilon, \quad \sum \delta^-(t_i) \geq \delta^-(t) - \varepsilon. \]

From Theorem 4 it follows from this that
\[ \mu_{S_1}^\delta = \mu_{S_2}^\delta, \quad \mu_{S_1}^{\delta^+} = \mu_{S_2}^{\delta^+}. \]
In other words, the following lemma is valid.

**Lemma 7.** For any open $G$
\begin{equation}
\omega^\pm(G) = \sup_{\{t\} \subset G} \sum \delta^\pm(t_i),
\end{equation}
where $\{t\}$ is a finite choice of nonoverlapping reduced geodesic triangles whose sectors at vertices are convex relative to the boundary.

**Lemma 8.** If the angle $\bar{\alpha}_1$ of a convex sector differs from the angle $\alpha_1$ between its sides, then
\[ 0 < \bar{\alpha}_1 - \alpha_1 \leq \theta - 2\pi, \]
where $\theta$ is the complete angle at the vertex of the sector.
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Proof. For a convex sector $\alpha_1 = \min\{\pi, \bar{\alpha}_1\}$. Therefore if $\alpha_1 \neq \bar{\alpha}_1$ we have $\alpha_1 = \pi$. Moreover, always $\alpha_1 = \min\{\pi, \bar{\alpha}_1, \bar{\alpha}_2\}$, where $\bar{\alpha}_2 = \theta - \bar{\alpha}_1$. Therefore in our case $\bar{\alpha}_2 \geq \pi$, and we have:

$$0 < \bar{\alpha}_1 - \alpha_1 = \bar{\alpha}_1 - \pi \leq \bar{\alpha}_1 + \bar{\alpha}_2 - 2\pi = \theta - 2\pi.$$ 

The lemma is proved.

Suppose that $A$ is a fixed point, $U$ some neighborhood of it, and $T$ any reduced triangle with the vertex $A$ lying in $U$ and having the following peculiarities: in $T$ the angle $\alpha$ at the vertex $A$ satisfies

$$\alpha = \bar{\alpha} < \pi,$$

and the sectors at the remaining vertices $B$ and $C$ of $T$ are convex relative to the boundary.

**Lemma 9.** The absolute value of the excess $\delta(T)$ of such a triangle $T$ will tend to zero when the neighborhood $U$ closes down on the point $A$.

**Proof.** 1. Suppose $\varepsilon > 0$. Choose a neighborhood $U$ of $A$ so small that the following conditions are satisfied.

1) The absolute curvature of $U$ with $A$ deleted is small:

$$\Omega(U - A) < \varepsilon.$$ 

This condition will be satisfied for a sufficiently small neighborhood $U$, since $\Omega$ is a completely additive function, and the sets $U_n - A$ form, as the radii $r_n$ of the neighborhoods $U_n$ tend to zero, an exhaustive sequence.

2) For any point of $U - A$, for the complete angle $\theta$

$$|2\pi - \theta| < \varepsilon < \pi.$$ 

This condition will be satisfied for a sufficiently small neighborhood $U$, since in any compact neighborhood $V$ of the point $A$ the sum $\sum_{i=1}^{\infty}|2\pi - \theta_i|$ is bounded and the basic portion of the points for which $\theta \neq 2\pi$ remains outside $U - A$, given only that the neighborhood $U$ is sufficiently small.

3) For any pair of shortest arcs issuing from $A$ and point $B, C$ on them, the angle $\alpha_0$ at the vertex $A$ in the plane triangle with the sides $AB, AC, BC$ differs little from the angle $\alpha$ between the shortest arcs:

$$|\alpha - \alpha_0| < \varepsilon.$$ 

This condition is satisfied in view of Theorem 11 of Chapter IV.

2. Consider a triangle $T$ meeting the requirements of Lemma 9 and lying in $U$. Suppose that $\bar{\alpha}, \bar{\beta},$ and $\bar{\gamma}$ are the sector angles at its vertices $A, B, C$. Mark on the side $AB$ a point $A'$ and join it to $C$ by a relative shortest arc $A'C$ lying in the triangle $ABC$. We take the point $A'$ so
close to $A$ that in the plane triangles with sides equal in length to $AB$, $AC$, $BC$ and $A'B$, $A'C$, $BC$ the corresponding angles $\alpha_0$, $\beta_0$, $\gamma_0$ and $\alpha'_0$, $\beta'_0$, $\gamma'_0$ will be almost the same:

$$ |\alpha_0 - \alpha'_0| < \varepsilon, \ |\beta_0 - \beta'_0| < \varepsilon, \ |\gamma_0 - \gamma'_0| < \varepsilon. $$

Since $\bar{\alpha} < \pi$, and the sectors $B$ and $C$ are convex relative to the boundary, we may suppose that the relative shortest arc $A'C$ excises from $ABC$ a reduced geodesic triangle $A'BC$ with sectors convex relative to the boundary at the vertices. All of this triangle lies in $U - A$.

3. For the triangle $A'BC$ itself and any reduced triangle $t$ excised from it by relative shortest arcs the absolute value of the excess, $\bar{\delta}(t)$, is small in view of equation (29) and Lemma 7:

$$ |\bar{\delta}(t)| < \varepsilon. $$

All the more, for the angles between the sides of such triangles

$$ \delta(t) < \varepsilon. $$

In view of the convexity of the sectors of these triangles we may suppose that the angles of these triangles are measured in the space $R'$ represented by the triangle $A'BC$, distinguished from the enveloping manifold.

By the fundamental theorem on angles of a triangle, applied to the space $R'$, we conclude from (34) that

$$ \alpha' - \alpha'_0 < \varepsilon, \ \beta - \beta'_0 < \varepsilon, \ \gamma' - \gamma'_0 < \varepsilon, $$

where $\alpha'$, $\beta$, $\gamma'$ are the angles between the sides of the triangle $A'BC$.

From Lemma 8 and the estimate (30), and also the estimate (31), we therefore conclude that the analogous inequalities for the sector angles hold:

$$ \bar{\alpha}' - \alpha_0 < 3\varepsilon, \ \bar{\beta} - \beta_0 < 3\varepsilon, \ \bar{\gamma}' - \gamma_0 < 3\varepsilon. $$

4. From inequality (33) it follows in particular that

$$ \pi - \bar{\alpha}' - \bar{\gamma}' - \bar{\beta} < \varepsilon. $$

Combining this inequality with

$$ \bar{\alpha}' - \alpha_0 < 3\varepsilon, \ \bar{\gamma}' - \gamma_0 < 3\varepsilon, \ \alpha_0 + \beta_0 + \gamma_0 = \pi, $$

we obtain $\beta_0 - \bar{\beta} < 7\varepsilon$. Finally

$$ -3\varepsilon < \beta_0 - \bar{\beta} < 7\varepsilon. $$

Analogously
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Along with inequality (31) and equation (28) this gives

\[ |\delta(T)| = |\alpha + \beta + \gamma - (\alpha_0 + \beta_0 + \gamma_0)| < 15\varepsilon, \]

which, in view of the arbitrary smallness of \( \varepsilon > 0 \), proves Lemma 9.

**Lemma 10.** Suppose that \( \theta \) is the complete angle around a fixed point \( A \). Then for any \( \varepsilon > 0 \) we may encircle the point \( A \) by an arbitrarily small polygon \( Q \) which is convex relative to the boundary and with vertices at points through which there pass shortest arcs, decomposed by diagonals into reduced triangles \( t \) for which

\[ \left| \sum \delta(t_i) - (2\pi - \theta) \right| < \varepsilon. \]

**Proof.** Suppose that \( \theta > 0 \). We divide the angle around \( A \) by shortest arcs \( L_i \) \((i=1, \ldots, n)\) into sectors with angles less than \( \theta/2 \) and \( \pi \). Then we enclose \( A \) in a very small polygon \( Q \) with vertices on the \( L_i \). We subdivide the polygon \( Q \) by shortest arcs \( L_i \) into triangles \( t'_i \). Suppose that \( \alpha_i \) are the angles of its sectors at the vertex \( A \), and \( \beta'_i, \beta''_i \) the angles of the other two sectors of \( t'_i \). From Lemma 9 the excesses of the triangles \( t'_i \) may be supposed arbitrarily small in sum:

\[ \sum |\delta(t'_i)| < \varepsilon. \]

We decompose \( Q \) into reduced triangles \( t_i \) by diagonals proceeding from one vertex. Suppose that \( \beta_i \) are the angles of the sectors of these triangles. Evidently \( \sum \beta_i = \sum \beta'_i + \sum \beta''_i \). Therefore we have

\[ \sum \delta(t_i) = \sum \beta_i - (n - 2)\pi = \left( \sum \beta'_i + \sum \beta''_i + \sum \alpha_i - n\pi \right) + 2\pi - \sum \alpha_i = \sum \delta(t'_i) + 2\pi - \theta. \]

From inequality (37), inequality (36) therefore follows.

If \( \theta = 0 \), we may take as \( Q \) a two-gon, enclosing the point \( A \) (see Lemma 17 of Chapter IV), and the remainder of the proof remains the same as when \( \theta > 0 \).

Lemma 10 is proved.

**Lemma 11.** Suppose that a shortest arc \( L \) passes through the point \( A \), on one side of which the sector angle at the vertex \( A \) is equal to \( \beta \). Then for any \( \varepsilon > 0 \) there exists a semineighborhood of the point \( A \) lying on the same side of \( L \), in the form of a polygon \( Q \) convex relative to the boundary and with vertices at points through which there pass shortest arcs, with
one of the sides of $Q$ going along $L$ and containing inside itself the point $A$, and the polygon $Q$ decomposes by diagonals into reduced triangles $t$ for which

$$(38) \quad \left| \sum \delta(t_i) - (\pi - \beta) \right| < \varepsilon.$$ 

This lemma is proved analogously to Lemma 10.

7. Definition of $\omega^+$, $\omega^-$ using different triangles. Theorem 4 makes it possible to establish the equivalence of a series of definitions of the measure $\mu_{S,\phi}$ for various collections $S$ and functions $\phi$. We shall see how arbitrary may be the families of triangles which may serve as the basis for the definitions (18) and (19) without changing the resulting values of $\omega^+$ and $\omega^-$.

**Lemma 12.** The values of $\omega^+$ and $\omega^-$ do not change if in the definitions (18), (19), the $t \in S_1$ (i.e., reduced triangles convex relative to the boundary) are replaced by triangles of a substantially narrower class, namely arbitrary reduced geodesic triangles.

**Proof.** As in the proof of Lemma 6, we decompose $t$ into fine triangles $t_i$ so that the condition

$$\left| \sum \delta(t_i) - \delta(t) \right| < \varepsilon$$

is satisfied.

Here the triangles $t_i$ not adjoining the vertices of $t$ may be considered as belonging to $S_1$ and the $t_i$ adjoining the vertices of $t$ as chosen similarly to the triangles $t'_i$ in the proof of Lemma 10. The excesses of these triangles turn out in sum to be small in absolute value, and, excluding these triangles from the number of all the $t_i$, we obtain a collection of nonoverlapping $t_i \in S_1$ lying in $t$ for which

$$\left| \sum \delta(t_i) - \delta(t) \right| < 2\varepsilon.$$ 

The validity of Lemma 12 now follows from Theorem 4.

**Lemma 13.** The value of $\omega^+$ does not change if in the definition (18) the $t \in S_1$ are replaced by the triangles of a significantly narrower class: namely, simple (homeomorphic to a disc and convex relative to the boundary) triangles with vertices at points through which there pass shortest arcs. At the same time the excesses $\delta(t)$ over the sector angles may be replaced by the excesses $\delta(t)$ over the angles between the sides of the triangles.

**Proof.** Take a $t \in S_1$. The excess $\delta(t)$ does not decrease if we discard
exterior tails of the triangle in the case when they are present. The remaining triangle \( t' \), as in the proof of Lemma 9 of Chapter III, may be decomposed into triangles \( t_i \) convex relative to the boundary and homeomorphic to the disc, such that the condition

\[
\sum \delta(t_i) \geq \delta(t') - \varepsilon
\]

holds, and all \( t_i \), other than those adjacent to vertices of the triangle \( t' \), will have vertices at points through which there pass shortest arcs. For the same \( t_i \) which adjoin the vertices of \( t' \), as in the preceding lemma, the excesses \( \delta(t_i) \), and therefore the excesses \( \delta(t) \), may be made very small in sum. Excluding such triangles from the total number of triangles \( t_i \), we obtain a system of nonoverlapping triangles \( t_i \) contained in \( t \) and convex relative to the boundary, with vertices at points through which there pass shortest arcs, while

\[
\sum \delta^+(t_i) > \delta^+(t) - 2\varepsilon.
\]

From Theorem 4, Lemma 13 therefore follows.

**Lemma 14.** The value of \( \omega^- \) does not change if in the definition (20) we narrow down somewhat the class \( S \), by restricting ourselves to reduced triangles convex relative to the boundary and with vertices at points through which there pass shortest arcs.

The proof of this lemma is quite analogous to the proof of Lemma 12. Combining Lemmas 12—14, we may assert the following.

**Theorem 6.** In the definition

\[
\omega^+(G) = \sup \sum_{(t) \subset G, t_i \in (t)} \delta^+(t_i)
\]

we may without changing the value of \( \omega^+(G) \) consider systems of nonoverlapping triangles \( t_i \) from any class \( S \), given only that \( S \) contains all simple triangles with vertices at points through which there pass shortest arcs, and \( S \) is contained in the class of reduced geodesic triangles. Moreover, instead of \( \delta^+(t) \) one may take \( \delta^+(t) \).

In the definition

\[
\omega^-(G) = \sup \sum_{(t) \subset G, t_i \in (t)} \delta^-(t_i)
\]

one may without changing the value of \( \omega^-(G) \) consider systems of nonoverlapping triangles \( t_i \) from any class \( S \), given only that \( S \) contains the reduced triangles convex relative to the boundary with vertices at points through
which there pass shortest arcs, and $S$ is contained in the class of reduced geodesic triangles.

Remarks. 1) In definition (20) for $\omega^{-}(G)$ we cannot get along solely with triangles homeomorphic to the disc. For example, in a cone with complete angle $5\pi$ at the vertex $O$ the negative part of the curvature $\omega^{-} = 3\pi$ is concentrated at the vertex. But in the neighborhood of the vertex it is not possible to lay off triangles homeomorphic to the disc which have the sum of the excesses close to $-3\pi$.

2) For a convex triangle the excesses $\delta(t)$ and $\delta(t)$ 

Figure 69.

8. Absolute curvature.

Lemma 15. For each open set $G$ and for any $\varepsilon > 0$, there exists a finite set contained in $G$ of nonoverlapping reduced triangles $t$, convex relative to the boundary and with vertices at points through which there pass shortest arcs, for which the following two inequalities are satisfied simultaneously:

\begin{equation}
\sum \delta^+(t_i) > \omega^+(G) - 4\varepsilon, \quad \sum \delta^-(t_i) > \omega^-(G) - 4\varepsilon.
\end{equation}

If $\omega^+(G)$ or $\omega^{-}(G) = \infty$, we replace the corresponding expression $\omega^+(G) - \varepsilon$, $\omega^-(G) - \varepsilon$ by arbitrarily large $N$.

We shall for definiteness carry out the proof for the case of finite $\omega^+(G)$, $\omega^{-}(G)$. Moreover, we shall suppose that $\omega^+(G) \neq 0$, $\omega^{-}(G) \neq 0$. In the contrary case the assertion of the lemma is trivial.

In $G$ there exists a family of nonoverlapping reduced triangles $T_i^+$ (with vertices at points through which there pass shortest arcs), for which $\delta(T_i^+) > 0$ and $\sum \delta^+(T_i^+) > \omega^+(G) - \varepsilon$. Analogously there exists a collection of triangles $T_i^-$ with negative excesses for which $\sum \delta(T_i^-) > \omega^-(G) - \varepsilon$.

By partial change of multiply intersecting shortest arcs, we may arrange things so that all the intersections $T_i^+T_i^-$ form along with the remainders...
OTHER DEFINITIONS OF CURVATURE

\( T^+_i \cup T^-_j \), \( T^-_j \cup T^+_i \) a finite number of polygons. Moreover, this may be done in such a way that the position of the vertices and the values of the excesses of the triangles \( T^+_i \), \( T^-_j \) do not change and that the mutual nonoverlapping of the triangles inside the systems \( \{ T^+_i \} \), \( \{ T^-_j \} \) is preserved.

We decompose the entire system of polygons thus formed into fine nonoverlapping triangles \( t_k \) such that for the new vertices of the subdivisions \( \sum |2\pi - \theta| < \varepsilon \). This is achieved as in the proof of Lemma 6. Moreover, we may suppose that all the \( t_k \) are convex relative to the boundary and have vertices at points through which there pass shortest arcs. Finally, we may suppose that for all triangles \( t_k \) adjacent to the vertices of all the \( P_n \), the excesses are small: \( \sum |\tilde{\delta}(t_k)| < \varepsilon \). This last is achieved in the same way as in the proof of Lemma 12 of this chapter.

Consider a separate triangle \( T^+_i \). For the triangles \( t_k \) into which it is subdivided, as in formula (22), we have:

\[
\sum \tilde{\delta}(t_k) = I + II + \tilde{\delta}^+(T^+_i).
\]

In the case at hand \( I = \sum (\sum \tilde{\alpha}_i - 2\pi) \) consists of terms \( (\sum \tilde{\alpha}_i - 2\pi) \), referring to the new vertices of the subdivision. These terms are small in the absolute value of their sum. Moreover, there are terms \( (\sum \tilde{\alpha}_i - 2\pi) \), referring to the vertices of \( P_n \). Each of these vertices lay before on the side or at a vertex of one of the triangles \( T^+_i \), \( T^-_j \) and therefore was a point through which there pass shortest arcs. For it \( \sum \tilde{\alpha}_i - 2\pi \geq 0 \). Finally, in the sum \( II = \sum (\sum \tilde{\beta}_m - \pi) \) all the terms \( \sum \tilde{\beta}_m - \pi \geq 0 \).

Thus for all the \( t_k \) which make up the triangle \( T^+_i \), we obtain

\[
\sum \tilde{\delta}(t_k) > \sum \tilde{\delta}^+(T^+_i) - \varepsilon,
\]

so that for all general \( t_k \)

\[
\sum \tilde{\delta}^+(t_k) > \sum \tilde{\delta}^+(T^+_i) - \varepsilon.
\]

If we exclude from the family of triangles \( t_k \) all those adjacent to vertices of \( P_n \), then for the remaining \( t_k \) we will have

\[
(40) \sum \tilde{\delta}^+(t_k) > \sum \tilde{\delta}^+(T^+_i) - 2\varepsilon.
\]

Inequality (40) is preserved if we later supplement the system of triangles \( t_k \) thus selected by further triangles so that along with them there is no overlapping of one with one another.

Now we consider separately a triangle \( T^-_j \). For the triangles \( t_k \) into which it is subdivided,
\[
\sum \delta(t_k) = I + II + \delta(T_j) = I + II - \delta^-(T_j).
\]

The terms \(\sum \alpha_i - 2\pi, \sum \beta_m - \pi\) in the sums I, II refer to the new vertices of the subdivision, distinct from the vertices of \(P_n\), and once again are small in the sum of their absolute values.

We shall sum equations (41) over all \(T_j\). We shall drop all the terms of I and II referring to the new vertices of the subdivisions, interchange the signs on both sides of the equation and carry over the remaining sums I', II' to the left. We get

\[-\sum \delta(t_k) + I' + II' > \sum \delta^-(T_j) - \varepsilon.\]

Here I' and II' are the sums \(\sum (\sum \alpha_i - 2\pi), \sum (\sum \beta_m - \pi)\), extended over the vertices of \(P_n\) appearing in at least one \(T_j\).

The inequality evidently remains valid if we replace all the \(-\delta(t_k)\) by \(\delta^-(t_k)\) and extend \(\sum \delta(t_k)\) over all \(t_k\) and not just those from which \(T_j\) was formed, and in the sums I', II' we will consider all angles \(\alpha_i, \beta_m\) as sector angles, without replacing some of these by the smaller value \(\pi\) as was done in equation (24).

Moreover, the left side decreases by not more than \(\varepsilon\), if in the sum \(\sum \delta^-(t_k)\) we drop all the \(t_k\) adjacent to the vertices of \(P_n\).

After these transformations we will have

\[\sum \delta^-(t_k) + I'' + II'' > \sum \delta^-(T_j) - 2\varepsilon.\]

Here the sum \(\sum \delta^-(t_k)\) is extended over the same \(t_k\) as in inequality (40).

All the \(t_k\) retained are not adjacent to the vertices of \(P_n\), to which the sums I'' and II'' refer. By Lemmas 10 and 11, every term of the last sum may be, with a small error, replaced by the sum \(-\sum \delta(t_p)\) for non-overlapping triangles lying in a small neighborhood of the vertex in question. Carrying out such a substitution for all the vertices and decreasing the entire sum \(-\sum \delta(t_p)\) to the value \(\sum \delta^-(t_p)\), we finally obtain

\[(42) \sum \delta^-(t_k) + \sum \delta^-(t_p) > \sum \delta^-(T_j) - 3\varepsilon.\]

For the system of earlier chosen \(t_k\), with the addition of the \(t_i\) just chosen, from inequalities (40) and (42) and the choice of the systems \(T_i^+, T_j^-\), both inequalities (39) will be satisfied.

Lemma 15 is proved.

**Theorem 7.** The absolute curvature of an open set \(G\) is equal to

\[(43) \Omega(G) = \sup_{\{t\} \subseteq \Gamma} \sum_{t_i \in \{t\}} |\delta(t_i)|.\]
where the \{t\} are finite sets of nonoverlapping triangles, which may be drawn from various classes, beginning with the narrower class of reduced triangles convex relative to the boundary with vertices at points through which there pass shortest arcs, and ending with the wider class of arbitrary reduced geodesic triangles.

The validity of Theorem 7 follows from definition (26) and Lemma 15.

9. Curvature of a one-point set. We shall use the following notation:

\[ \delta(A) = 2\pi - \theta, \quad \delta^+(A) = \max\{0, \delta(A)\}, \]
\[ \delta^-(A) = -\min\{0, \delta(A)\}, \]

where \( \theta \) is the complete angle around the point \( A \).

Suppose that \( S \) is a system consisting of all reduced triangles and all points of the space in question. We understand nonoverlapping of triangles in the previous sense. Two points are considered to be nonoverlapping when they are distinct. A point and a triangle are considered nonoverlapping if the point lies outside of the triangle or is a vertex of it. The nonoverlapping thus defined evidently satisfies the conditions of subsection 1.

**Lemma 16.** For every open set \( G \)

\[ \omega^\pm(G) = \sup_{\{t,A\} \subseteq G} \left[ \sum \delta^\pm(t_i) + \sum A_j \delta^\pm(A_j) \right], \]

where the \{\( t, A \)\} are finite systems of pairwise nonoverlapping triangles \( t \) and points \( A \) of \( S \).

**Proof.** Evidently

\[ \omega^+(G) \leq \sup_{\{t,A\} \subseteq G} \left[ \sum \delta^+(t_i) + \sum \delta^+(A_j) \right], \]

so that we need only prove the reverse inequality.

Suppose that \{\( t, A \)\} is any finite system of nonoverlapping triangles and points of \( S \). Each triangle \( t_i \subseteq \{t,A\} \) with a positive excess may be decomposed into fine triangles \( t_i^k \) and those discarded which are adjacent to the vertices of \( t_i \), this being done in such a way that the condition

\[ \sum_i \left| \sum_k \delta(t_i^k) - \delta(t_i) \right| < \varepsilon \]

is maintained.

Therefore

\[ \sum_{i,k} \delta^+(t_i^k) \geq \sum \delta^+(t_i) - \varepsilon. \]

Now all the points \( A \) lie isolated from all the \( t_i^k \). Each point \( A_i \), with
a positive excess may, by Lemma 10, be encircled by a very small polygon $Q_j$ subdivided into reduced triangles $t_i^j$ in such a way that

$$\sum_j |\sum_k \delta(t_i^j) - \delta(A_j)| < \varepsilon.$$ 

Therefore

$$\sum_{j,k} \delta^+(t_i^j) \geq \sum_j \delta^+(A_j) - \varepsilon.$$

For the system $t^k$ of all triangles $t_i^k$, we will have

$$\sum_k \delta^+(t^k) \geq \sum \delta^+(t_i) + \sum \delta^+(A_j) - 2\varepsilon,$$

where the sum on the right is extended over all $t_i, A_j \in \{t,A\}$.

From the inequality just obtained it follows that

$$\omega^+(G) \geq \sum \delta^+(t_i) + \sum \delta^+(A_j) - 2\varepsilon.$$

Because of the arbitrary smallness of $\varepsilon > 0$ and the arbitrariness in the choice of the system $\{t,A\} \subset G$ the inequality inverse to (45) therefore follows. Along with (45) it yields equation (44).

The proof for $\omega^-(G)$ is quite analogous.

**Lemma 17.** For every open set $G$

(46) \[ \Omega(G) = \sup \left\{ \sum_{t_i \in [t,A]} |\delta(t_i)| + \sum_{A_j \in [t,A]} |\delta(A_j)| \right\}, \]

where $\{t,A\}$ are finite systems of pairwise nonoverlapping reduced triangles and points.

**Proof.** The fact that the left side of equation (46) does not exceed the right follows from Theorem 7. The reverse relation is proved as in Lemma 16.

**Theorem 8.** For a point $A$ with a complete angle $\theta$,

(47) \[ \omega^+(A) = (2\pi - \theta)^+, \quad \omega^-(A) = (2\pi - \theta)^-; \]

(48) \[ \omega(A) = 2\pi - \theta, \quad \Omega(A) = |2\pi - \theta|, \]

where by definition we have written

$$ (2\pi - \theta)^+ = \max \{0, 2\pi - \theta\}, \quad (2\pi - \theta)^- = \max \{0, \theta - 2\pi\}. $$

**Proof.** It follows from Lemma 16 that

(49) \[ \omega^\pm(A) = \inf_{G \ni A} \omega^\pm(G) \geq (2\pi - \theta)^\pm. \]

We shall show that at the same time $\omega^\pm(A) \leq (2\pi - \theta)^\pm$. For definiteness
we consider \( \omega^-(A) \). If \( \omega^-(A) = 0 \), the inequality in question is evident. Suppose that \( \omega^-(A) > 0 \). Select \( \varepsilon > 0 \), \( \varepsilon < (1/2)\omega^-(A) \). Select a neighborhood \( U \) of the point \( A \) so small that

\[
\Omega(U - A) < \varepsilon, \quad \sum (2\pi - \theta_i) < \varepsilon,
\]

where \( \Omega(U - A) \) is the absolute curvature of the neighborhood with the point \( A \) deleted, and \( \theta_i \) are the complete angles around arbitrary points of \( U - A \).

In the neighborhood \( U \) there exists a system of nonoverlapping reduced triangles \( T_i \) with negative excesses, for which

\[
-\sum \delta(T_i) > \omega^-(U) - \varepsilon.
\]

The point \( A \) lies in at least one of the \( T_i \). Otherwise all the \( T_i \subset U - A \) and \( -\sum \delta(T_i) \leq \Omega(U - A) < \varepsilon < \omega^-(U) - \varepsilon \).

Suppose that \( A \) lies inside one of the triangles \( T_i \), say inside \( T_i^0 \). We encircle the point \( A \) by a polygon \( P \) lying inside \( T_i^0 \) and subdivide \( T_i^0 \) into nonoverlapping reduced triangles \( t_i \), counting among them the triangles \( t_i' \) obtained on the subdivision of \( P \) by diagonals. By Lemma 10, we may suppose that \( P \) is chosen so that

\[
\sum \delta(t_i') < (2\pi - \theta) < \varepsilon.
\]

From condition (5) and a relation of type (21) it follows that

\[
\left| \sum \delta(t_i') - (2\pi - \theta) \right| < \varepsilon.
\]

Therefore for \( t_i \) and all the \( T_i \) other than \( T_i^0 \) we will have

\[
-\sum \delta(T_i) - \sum \delta(t_i') \geq \omega^-(U) - 2\varepsilon.
\]

On the left side of this last inequality we drop all terms other than those referring to the \( t_i' \). From the first of conditions (50), the excesses of the rejected triangles are in absolute value less than \( \varepsilon \), so that

\[
-\sum \delta(t_i') > \omega^-(U) - 3\varepsilon.
\]

Along with inequality (52) this yields

\[
(2\pi - \theta) - \sum \delta(t_i') - \varepsilon > \omega^-(U) - 4\varepsilon \geq \omega^-(A) - 4\varepsilon.
\]

If the point \( A \) does not lie inside one of the \( T_i \), then it lies on the boundary of one or several \( T_i \). We then proceed as follows. We divide each triangle adjacent to \( A \) at a vertex into finite triangles \( t_i \). The sum of their excesses, because of the second of the inequalities (50), will be only little different
from the excesses of $T_i$. In each triangle $T_i$ adjacent to $A$ along a side, we encircle the point $A$ by a semineighborhood in the form of a polygon $P$, subdivided by diagonals into triangles $t_i$. From Lemma 11, one may do this in such a way that the inequality

$$\left| \sum \delta(t_i') - (\pi - \tilde{\beta}) \right| < \varepsilon$$

is satisfied, where $\tilde{\beta}$ is the sector angle in the triangle at the vertex $A$. Moreover we may suppose that $T_i$ is subdivided into reduced nonoverlapping triangles $t_i$, including the $t_i'$.

For all $t_i$ obtained on the subdivision of $T_i$ adjacent to $A$, and for the remaining nonsubdivided $T_i$ we shall have

$$-\sum \delta(t_i') - \sum \tilde{\delta}(T_i) > \omega^{-}(U) - 2\varepsilon.$$

Dropping on the left all terms other than those relating to the triangles $t_i'$, we obtain

$$-\sum \tilde{\delta}(t_i') > \omega^{-}(U) - 3\varepsilon.$$

Taking into account inequality (54), we again arrive at the inequality

$$(2\pi - \theta)^- \geq \omega^{-}(U) - 4\varepsilon \geq \omega^{-}(A) - 4\varepsilon.$$

Because of the arbitrary smallness of $\varepsilon > 0$ it therefore follows that $(2\pi - \theta)^- \geq \omega^{-}(A)$, which along with inequality (49) yields

$$\omega^{-}(A) = (2\pi - \theta)^-.$$

For $\omega^{+}(A)$ the proof is analogous. We shall not carry it out.

Equations (48) follow immediately from (47). Theorem 8 is completely established.

10. Definition of the curvature using polygons. Recall that by a polygon we understand a compact set whose boundary consists of a finite number of simple closed broken curves. We shall this time waive the requirement of connectedness. If the space is a closed two-dimensional manifold, then the entire space will be regarded as a polygon. The number of boundary curves in this case is equal to zero. The vertices of its broken lines will be called the vertices of the polygon.

The excess $\delta(P)$ of the polygon will be the quantity

$$\delta(P) = 2\pi \chi - \sum (\pi - \tilde{\alpha}_i),$$

where $\chi$ is the Euler characteristic of the polygon $P$ and the $\tilde{\alpha}_i$ the angles of the sectors adjacent to the polygon $P$ at its vertices.
The quantity \( \delta(P) \) is inconvenient in that it depends on the choice of vertices. On the sides of \( P \) there may be points at which the sector angle on the side of the polygon is larger than \( \pi \). It is sufficient to add such a point to the vertices of \( P \) to make the quantity \( \delta(P) \) change. We introduce therefore a somewhat different characteristic,

\[
\delta\hat{}(P) = 2\pi \chi - \sum (\pi - \alpha_i),
\]

where the sum \( \sum (\pi - \alpha_i) \) is extended over all the points of the boundary of \( P \) at which the sector angle \( \alpha_i \) on the side of \( P \) differs from \( \pi \). There are not more than countably many such points and the sum \( \sum (\pi - \alpha_i) \) is finite. It consists of a finite number of terms relating to vertices and terms relating to points on the sides of \( P \). At the latter \( |\pi - \alpha_i| < |2\pi - \theta_i| \), and the sum \( \sum |2\pi - \theta_i| \) is finite because of the compactness of \( P \).

**Lemma 18.** The quantity \( \delta\hat{}(P) \) is additive in the following sense. If we subdivide the polygon \( P \) into a finite number of polygons \( P_i \), homeomorphic to the disc, then

\[
\delta\hat{}(P) = \sum_{i=1}^{n} \delta\hat{}(P_i) + \sum_{i=1}^{\infty} (2\pi - \theta_i),
\]

where \( \theta_i \) are the complete angles around points on the boundary of \( P \).

This lemma follows from the definition of \( \delta\hat{} \) and Euler's theorem.

We shall not prove the following two assertions, which are obvious after the material presented in this section.

1. For every polygon \( P \)

\[
0 \leq \delta\hat{}(P) - \delta(P) = - \sum (\pi - \beta_i) \leq \sum (2\pi - \theta_i),
\]

where \( \beta_i \equiv \pi \) are the sector angles at separate points on the sides of \( P \), and the \( \theta_i \) are the complete angles around the same points.

2. If in a reduced triangle \( T \) convex relative to the boundary we reject the exterior tails, if there are any, and for the triangle \( T' \) thus obtained consider the characteristic \( \delta\hat{}(T') \), then

\[
\delta\hat{}(T') = \delta(T) + \alpha + \sum (\beta_i - \pi),
\]

where \( \alpha \) is the sector angle in the triangle \( T' \) on the basis of the rejected tail, and \( \beta_i \equiv \pi \) are the sector angles for separate points on the sides of \( T' \).
Suppose that $S'$ is a system of all triangles homeomorphic to the disc and of separate points. We regard the triangles $t_1, t_2 \in S'$ as nonoverlapping if they have no common interior points. The points $A_1, A_2 \in S'$ are nonoverlapping if they are distinct. A point $A$ and the triangle $t \in S'$ are nonoverlapping if $A$ does not lie inside $t$. The difference from the system $S$ appearing in Lemma 16 is that we are considering only triangles homeomorphic to the disc and are now admitting points on the side of a triangle as nonoverlapping with the triangle, whereas before we admitted only points lying at a vertex or outside the triangle.

**Lemma 19.** For every open set $G$

$$\omega^\pm(G) = \sup \left[ \sum_{(t, A) \in G} \delta^\pm(t_i) + \sum_{A_j \in (t, A)} (2\pi - \theta_i) \right],$$

where $\{t, A\}$ are finite systems of pairwise nonoverlapping $t, A \in S'$ and the $\theta_i$ are the complete angles around the points $A_j$.

**Proof.** We denote the right side of (59) by $I^\pm$.

1. By Lemma 13, there is in $G$ a system of triangles homeomorphic to the disc for which $\sum \delta(t) > \omega^+(G) - \varepsilon$. For the same triangles $\sum \delta(t) > \omega^+(G) - \varepsilon$. Because of the arbitrariness of $\varepsilon > 0$ it therefore follows that

$$\omega^+(G) \leq I^+.$$

2. Suppose that $\{t, A\} \subseteq G$ is a finite system of pairwise nonoverlapping $t, A \in S'$ for which $\sum \delta(t) + \sum \delta(A) > I^+ - \varepsilon$. Consider a triangle $t$. On its sides there may be a point with sector angle $\beta \supseteq \pi$ on the side of the triangle. We mark on the sides of $t$ points $A \in \{t, A\}$ and points almost exhausting $\sum (\beta - \pi)$. Subdivide $t$ into reduced triangles $t_i$ with vertices at distinguished points and at other points through which there pass shortest arcs. As in relation (22), we will have

$$\sum \delta(t_i) = \delta(t) + \sum (\sum \alpha - 2\pi) + \sum (\beta - \pi).$$

But by construction $\sum \alpha - 2\pi \geq 0$, $\delta(t) + \sum (\beta - \pi) \approx \delta(t)$. Therefore

$$\sum \delta(t_i) + \sum \delta(A) > I^+ - 2\varepsilon.$$

Because of the arbitrary smallness of $\varepsilon > 0$, and from Lemma 16, it therefore follows that

$$\omega^+(G) \geq I^+.$$

Along with inequality (60) this yields $\omega^+(G) = I^+.$

3. By Lemma 16, there exists a finite system $\{t, A\} \subseteq G$ of nonoverlapping $t, A \in S$, for which $-\sum \delta(t) - \sum \delta(A) > \omega^-(G) - \varepsilon$. If $t$ has external tails,
we remove them, and to the collection of points \( A \) we add the new vertices of the resulting triangle \( t' \). On the sides of \( t' \) we mark off points almost exhausting \( \sum (\bar{\delta} - \pi) \), and also add them to the collection of points \( A \). For the resulting system of triangles \( t' \) and points \( A' \), it follows from equation (58) that

\[
\sum \bar{\delta}(t) + \sum (2\pi - \theta)^- > \omega^-(G) - 2\varepsilon.
\]

Therefore

\[
(61) \quad I^- \geq \omega^-(G).
\]

4. Suppose \( \{t,A\} \subset G \) is a finite system of nonoverlapping \( t \), \( A \subset S' \), for which

\[-\sum \bar{\delta}(t) - \sum \delta(A) > I^- - \varepsilon.\]

On the sides of \( t \) we mark off points \( A \subset \{t,A\} \) and points almost exhausting \( \sum (\bar{\delta} - \pi) \). We decompose \( t \) into reduced triangles \( t \), with vertices at distinguished points and at points with the sum \( \sum |2\pi - \theta| \) small. As in equation (22), we will have

\[
\sum \bar{\delta}(t_i) = \bar{\delta}(t) + \sum (\sum \bar{\alpha} - 2\pi) + \sum (\bar{\delta} - \pi).
\]

This time \( \sum \bar{\alpha} - 2\pi \approx 0 \), \( \bar{\delta}(t) + \sum (\bar{\delta} - \pi) \approx \bar{\delta}(t) \), so that

\[-\sum \bar{\delta}(t_i) - \sum \delta(A) > I^- - 2\varepsilon.
\]

Because of the arbitrary smallness of \( \varepsilon > 0 \) and from Lemma 16, it therefore follows that \( \omega^-(G) \leq I^- \), which along with (61) yields \( \omega^-(G) = I^- \).

We have carried out the argument under the assumption that \( \omega^\pm(G) \) and \( I^\pm \) are finite. In the case of infinite values the proof remains the same as before, the difference being only that \( \omega^\pm(G) - \varepsilon \) and \( I^\pm - \varepsilon \) are replaced by arbitrarily large numbers.

Lemma 19 is completely proved.

We shall say that the polygon \( P' \) follows the polygon \( P \) if \( P \subset P' \).

The partially ordered set of polygons contained in an open set \( G \) forms a so-called directed set and one may consider the limits

\[
(62) \quad \lim_{P \subset G, P \to G} \bar{\delta}(P), \quad \lim_{P \subset G, P \to G} \bar{\delta}(P).
\]

**Theorem 9.** If for the open set \( G \) at least one of the quantities \( \omega^+(G) \), \( \omega^-(G) \) is finite, then there exists a finite or an infinite limit of definite sign for each of the quantities in (62), and the equation

\[
(63) \quad \omega(G) = \lim_{P \subset G, P \to G} \bar{\delta}(P) = \lim_{P \subset G, P \to G} \bar{\delta}(P)
\]

holds. If \( \omega^+(G) = \omega^-(G) = \infty \), the limits (62) do not exist.
Proof. 1. Suppose that $\omega^+(G)$ and $\omega^-(G)$ are finite. We choose a finite system $\{t\} \subset G$ of nonoverlapping triangles convex relative to the boundary, for which simultaneously

$$\sum \delta^+(t_i) > \omega^+(G) - \varepsilon, \quad \sum \delta^-(t_i) > \omega^-(G) - \varepsilon.$$ 

Such a system exists by Lemma 15. Moreover, we may suppose that two sides of any two triangles $t_i, t_j \in \{t\}$ touch along a finite number of intervals and points. This may be achieved by reducing multiply contacting pieces of shortest arcs to one segment.

The triangles of $\{t\}$ with positive excesses $\delta(t)$ will be denoted by $t^+_i$ and those with negative excesses by $t^-_i$.

We discard the external tails in the triangle $t$ and turn from $\delta(t)$ to $\delta(t)$. In view of (58) we will have

$$\sum \delta^+(t_i) \geq \sum \delta^+(t_i) > \omega^+(G) - \varepsilon,$$

$$-\sum \delta(t^-_i) = -\sum \delta(t^-_i) - \sum \bar{\alpha} - \sum (\bar{\beta} - \pi)$$

$$> \omega^-(G) - \varepsilon - \sum \bar{\alpha} - \sum (\bar{\beta} - \pi).$$

2. We encircle the system $\{t\}$ chosen above by a polygon $P \subset G$ and consider an arbitrary polygon $P' \supset P, P' \subset G$.

We decompose $P'$ into triangles homeomorphic to the disc, including among them all the $t^+_i$ and also some $t^-_i$. By Lemma 18

$$\hat{\delta}(P') = \sum \hat{\delta}(t^+_i) + \sum \hat{\delta}(t^-_i) + \sum (2\pi - \theta_i).$$

The last sum possibly consists of a countable number of terms. It certainly contains terms $2\pi - \theta_i$ corresponding to vertices of the angles $\bar{\alpha}$ and $\bar{\beta}$ figuring in inequality (65). If we select a finite number of terms from the sum on the right, then, from Lemma 19, the positive sums do not exceed $\omega^+(G)$, and the negative sums do not exceed, in absolute value, $\omega^-(G)$. At the same time only the sum $\sum \hat{\delta}(t^+_i)$, from (64), exceeds $\omega^+(G) - \varepsilon$, and the sum $-\sum (\hat{\delta}^-) - \sum (2\pi - \theta_i)$ from (65), exceeds $\omega^-(G) - \varepsilon$. Therefore

$$|\hat{\delta}(P') - \omega(G)| < \varepsilon,$$

i.e., for finite values of $\omega^+$ and $\omega^-$ there exists

$$\lim_{P \in G, P \rightarrow G} \delta(P) = \omega(G).$$

3. If $\omega^+(G) < \infty$ and $\omega^-(G) = \infty$, then in the same way we find that

$$\lim_{P \in G, P \rightarrow G} \hat{\delta}(P) = -\infty.$$ If $\omega^+(G) = \infty$ and $\omega^-(G) < \infty$ we obtain

$$\lim_{P \in G, P \rightarrow G} \delta(P) = +\infty.$$
4. We shall show the existence of a finite limit \( \lim_{P \subset G, P \to G} \hat{\delta}(P) \), or an infinite limit with a definite sign, implies that the limit \( \lim_{P \subset G, P \to G} \tilde{\delta}(P) \) exists and is equal to it. If the first limit is equal to \(-\infty\), then in the second limit we have also \(-\infty\), since \( \hat{\delta}(P) \geq \tilde{\delta}(P) \). Suppose that the first limit is not equal to \(-\infty\). Then \( \sum (2\pi - \theta)^- < \infty \) for any system of points of \( G - P \). Otherwise, for any fixed \( P \subset G \) we would have a system of points of \( G - P \) with an arbitrarily large sum \( \sum (2\pi - \theta)^- \). These points could be encircled, by Lemma 10, by nonoverlapping polygons \( P_i \) homeomorphic to the disc with an arbitrarily large sum \( \sum \hat{\delta}(P_i) \). For the polygon \( P' = P + \sum P_i \) the value of \( \hat{\delta}(P') \) would be strongly different from \( \hat{\delta}(P) \), which contradicts the existence of \( \lim_{P \subset G, P \to G} \hat{\delta}(P) \). Hence \( \sum (2\pi - \theta)^- < \infty \).

Mark off a finite number of points almost exhausting \( \sum (2\pi - \theta)^- \). For any polygon \( P \) enclosing these points and therefore not having any of them on its boundary, we will have \( |\hat{\delta}(P) - \tilde{\delta}(P)| < \varepsilon \). Therefore the following limits exist and are equal:

\[
\lim_{P \subset G, P \to G} \delta(P) = \lim_{P \subset G, P \to G} \hat{\delta}(P).
\]

5. Conversely, if \( \lim \tilde{\delta}(P) \) exists, then certainly \( \lim \hat{\delta}(P) \) exists, since \( \hat{\delta}(P_0) \) may be considered as the limit of \( \lim \tilde{\delta}(P_0) \) as \( P_0 \) is enriched by vertices.

In order to complete the proof of Theorem 9 it remains to be proved that from the existence of the limits (62) follows the boundedness of \( \omega^+(G) \) or \( \omega^-(G) \).

6. Suppose that

\[(66) \lim \hat{\delta}(P) = I < +\infty.\]

We shall prove that \( \omega^+(G) < \infty \).

Choose a polygon \( P \subset G \) such that for any \( P' \supset P, P' \subset G \)

\[(67) \hat{\delta}(P') < N,\]

where \( N \) is some finite number. The existence of such a \( P \) follows from (66).

If \( \omega^+(G) = \infty \), then in \( G - P \) there exists a finite system of nonoverlapping triangles \( t \) homeomorphic to the disc with vertices at points through which there pass shortest arcs, for which

\[\sum \hat{\delta}(t) > N - \hat{\delta}(P).\]

Moreover, these triangles may be taken to be adjacent along not more than a finite number of segments and points.
Suppose that to the triangle \( t \), at the point \( A \) on its side, we adjoin a second triangle. We encircle the point \( A \) in \( t \) by a semineighborhood in the form of an \((n+1)\)-gon, one of whose sides \( A'A'' \) goes along the side of \( t \). The notation used below is shown in Figure 70.

Instead of \( t \) consider the polygon \( P \), obtained by deleting the indicated semineighborhood from \( t \). We shall see how the excess \( \hat{\delta}(t) \) is changed.

In the definition
\[
\hat{\delta}(t) = \delta(t) - \sum (\pi - \beta)
\]
the terms \( \beta - \pi \) referring to the points of the segment \( A'A'' \) will be lost. All these terms may be taken to be small in their sum, except for the term \( \beta - \pi \) corresponding to the point \( A \). For this term we have
\[
\beta - \pi = (\beta + \sum \gamma' + \alpha_2 + \alpha_3 - n\pi) - \left[ \sum \gamma' + \sum \gamma - 2\pi(n-1) \right] - (\alpha_1 + \alpha_2 - \pi) - (\alpha_3 + \alpha_4 - \pi) + (\alpha_1 - \pi) + (\alpha_4 - \pi) + \sum (\gamma - \pi).
\]

But all the terms on the right except the last may be supposed arbitrarily small, if the construction is carried out in a small neighborhood of the point \( A \). Now the last term is exactly what is newly appearing in the definition of \( \hat{\delta}(P_i) \) in place of \( \beta - \pi \).

Therefore
\[
\hat{\delta}(P_i) \approx \hat{\delta}(t).
\]

Analogous excisions may be carried out for the vertices of \( t \) and for several points \( A \) on the boundary of \( t \).

The resulting polygons, homeomorphic to the disc, are isolated or adjoin one another along entire segments of the boundary, and for them \( \sum \hat{\delta}(P_i) > N - \hat{\delta}(P) \). For the polygon \( P' = P + \sum P_i \) we will have
\[
\hat{\delta}(P') = \hat{\delta}(P) + \sum \hat{\delta}(P_i) + \sum (2\pi - \theta),
\]
where \( |2\pi - \theta| < \epsilon \), since the \( \theta \) are the complete angles around points on the interior curves of the decomposition of \( \sum P_i \) into the pieces \( P_i \). Therefore from (68) we obtain
which contradicts the choice of $P$.

7. Suppose finally that $\lim_{P \subseteq G, P \rightarrow G} \hat{\delta}(P) = +\infty$. We shall show that $\omega^-(G) < \infty$. Choose a polygon $P \subseteq G$ so that for any $P' \supseteq P$, $P' \subseteq G$ we always have $\hat{\delta}(P') > 0$.

Suppose that $\omega^-(G) = \infty$. Then $\omega^-(G - P) = \infty$.

There exists a positive $C < \infty$ such that for any system of points in $G - P$

$$\sum (2\pi - \theta)^- < C,$$

for otherwise there would be a system of points with $\sum (2\pi - \theta)^+ > |\hat{\delta}(P)|$, and, enclosing them by small polygons $P_i$, we would find a polygon $P' = P + \sum P_i$ for which $\hat{\delta}(P') < 0$.

Choose in $G - P$ a system of nonoverlapping reduced triangles convex relative to the boundary, for which

$$-\sum \hat{\delta}(t) > |\hat{\delta}(P)| + 2C.$$

Discard the exterior tails from the triangles $t$. Then pass to $-\sum \hat{\delta}(t')$ for the resulting triangles $t'$. Then, from equation (58), we have lost in comparison with $-\sum \hat{\delta}(t)$ by not more than $C$.

As in subsection 6, we may, by very slight changes in $-\sum \hat{\delta}(t')$, create polygons $P_r$ which adjoin only along entire segments.

Turning to the excesses $-\hat{\delta}(P_r)$, we further decrease the sum by no more than $C$, since the change can only take place at the expense of the curvature of points appearing on interior curves of the net of the subdivision $\sum P_r$ into the $P_r$.

Finally, for $P' = P + \sum P_r$ we will have

$$\hat{\delta}(P') = \hat{\delta}(P) + \hat{\delta}(\sum P_r) < \hat{\delta}(P) - |\hat{\delta}(P)| \leq 0,$$

which contradicts the choice of $P$.

Theorem 9 is completely proved.

Open sets $G$ containing $M$ form a directed set if one regards $G'$ as following $G$ when $G' \subseteq G$.

**Lemma 20.** If for the set $M$ either the quantity $\omega(M)$ or $\lim_{G \supseteq M, G \rightarrow M} \omega(G)$ is defined, then the other is also defined and we have the equation

$$\omega(M) = \lim_{G \supseteq M, G \rightarrow M} \omega(G).$$

**Proof.** 1. Suppose that $\omega^+(M)$ and $\omega^-(M)$ exist and are finite. Choose
$G_1 \supset M$ and $G_2 \supset M$ for which
\[ \omega^+(G_1) < \omega^+(M) - \varepsilon, \quad \omega^-(G_2) > \omega^-(M) - \varepsilon. \]
For each $G \supset M$, $G \subset G_1$ we will have
\[ |\omega(G) - \omega(M)| < \varepsilon. \]
If only one of the quantities $\omega^\pm(G)$ is finite, then the proof is obtained analogously, namely by replacement of the corresponding quantity $\omega^\pm(G) - \varepsilon$ by arbitrarily large $N$.

2. If $\lim_{G \supset M, G \to M} \omega(G)$ exists, then, beginning with some $G$ the quantity $\omega(G)$ is defined, so that one of the quantities $\omega^\pm(G)$ is finite. Accordingly $\omega^+(M)$ or $\omega^-(M)$ is finite, and $\omega(M)$ has a meaning.

The lemma is proved. From Theorem 9 and Lemma 21 there follows a definition of $\omega(M)$ equivalent to the preceding one, avoiding the introduction of the concepts $\omega^\pm(M)$:
\[ \omega(G) = \lim_{P \in G, P \to G} \delta(P), \quad \omega(M) = \lim_{G \supset M, G \to M} \omega(G). \]

4. **Some preliminary estimates.** The Lemmas 21—23 which follow are used in Chapter VI. The estimates given in these lemmas are later substantially improved.

11. **Excess and curvature of a polygon.**

**Lemma 21.** Suppose that the vertices $A_i$ of a polygon $P$ lie at points $A_i$ where $\theta \neq 0$. For the excess $\delta(P)$ of a polygon $P$, defined by formula (55), we always have the estimate
\[ -\omega^-(P - \sum_i A_i) \leq \delta(P) \leq \omega^+(P - \sum_i A_i). \]

**Proof.** Encircle $P$ by an arbitrary open region $G$. For the polygon $P$ we have:
\[ \delta(P) = \delta(P) + \sum_j (\pi - \beta_j), \]
where $\beta_j$ are the sector angles not equal to $\pi$ on the side of $P$ at points on the sides of $P$ distinct from the vertices.

Decompose the sectors at each vertex $A_i$ into convex sectors smaller than $\pi$. From each of these sectors we excise by a shortest arc a very small convex triangle $t'_k$ homeomorphic to the disc. The remaining polygon $P - \sum_i t'_k$ lies inside $G - \sum A_i$. Decompose this polygon into convex triangles $t_i$. Then $P$ will be decomposed into pieces $t'_k, t_i$. From Lemma 18,
\[ \delta(P) = \sum_k \delta(t'_k) + \sum_i \delta(t_i) + \sum_m (2\pi - \theta_m), \]
where \( \theta_m \) are complete angles different from \( 2\pi \) around points on the curves and vertices of the subdivision not lying on the boundary of \( P \).

We may suppose that for all the triangles \( t_k' \) which adjoin the vertices \( A_i \), the following conditions are satisfied: 1) the excesses \( \hat{\delta}(t_k') \) differ by little even in sum from the excesses \( \hat{\delta}(t_k) \); 2) the excesses \( \hat{\delta}(t_k) \) differ by little from \( \delta(t_k) \); the excesses \( \delta(t_k) \) are small. The first condition may be guaranteed by making the negative part of the curvature small on the sides of \( t_k' \) close to the vertices \( A_i \), in view of the fact that it majorizes the difference between \( \sum \hat{\delta}(t_k) \) and \( \sum \hat{\delta}(t_k) \). The second condition is guaranteed by the fact that the decomposition of the sectors for the \( A_i \) was carried out into convex angles with sectors less than \( \pi \), so that for them the angle does not differ from the sector angle, and for the other angles of \( t_k' \) the angle and the sector angle can differ by no more than the negative curvature of the vertex, and the curvatures of the vertices are small in sum. Finally, the third condition may be guaranteed, for example, from Lemma 11 of Chapter II.

Thus we may suppose that

\[
\sum_k \hat{\delta}(t_k') = \epsilon v,
\]

where \( \epsilon > 0 \) is an arbitrarily small number and \( |v| < 1 \). Hence

\[
(70) \quad \delta(P) = \sum_i \hat{\delta}(t_i) + \sum_m (2\pi - \theta_m) + \sum_j (\pi - \beta_j) + \epsilon v.
\]

Taking into account the fact that all the triangles \( t_i \) and points with the angles \( \theta_m, \beta_j \) lie in the open set \( G - \sum A_i \), and also that each difference \( \pi - \beta_j \) forms only a portion of the negative curvature at the corresponding point and that \( \epsilon \) is arbitrarily small, we may on the basis of Lemma 19 and equation (70) conclude that

\[
-\omega^- (G - \sum A_i) \leq \delta(P) \leq \omega^+(G - \sum A_i).
\]

Because of the arbitrariness of \( G \supset P \), (69) therefore follows.

**Lemma 22.** Under the conditions of Lemma 21 every angle of the polygon \( P \) may be cut by a broken curve lying in it close to the vertex and such that the excess of the resulting polygon \( P' \) will differ by arbitrarily little from the excess of \( P \).

**Proof.** Suppose that the angle in the sector at the vertex \( A \) is cut by the broken line \( B_1B_2\cdots B_n \). With the notations of Figure 71, we will have
\[ \delta(P) - \delta(P') = \alpha - \pi - \sum_{i=1}^{n} (\tilde{\phi}_i - \pi) \]

\[ = \delta(Q) + (\tilde{\phi}_1 + \tilde{\phi}_n - \pi) + \sum_{i=2}^{n-1} \phi(B_i) + (\tilde{\phi}_n + \tilde{\phi}_1 - \pi). \]

Estimating the first term in this last sum by Lemma 21, and the remaining terms by the curvatures of the corresponding points, we have:

\[ |\delta(P) - \delta(P')| \leq 2\Omega(Q - A). \]

Therefore, if the polygon \( Q \) lies in a sufficiently small neighborhood of the point \( A \), then

\[ |\delta(P) - \delta(P')| < \varepsilon. \]

12. Comparison with a plane triangle.

Lemma 23. If the reduced triangle \( T \) lies in an open region \( G \), and if moreover each pair of points on the sides of \( T \) may be joined by a shortest arc, not leaving \( G \), then for each angle \( \alpha \) between the sides of \( T \) and the corresponding angle \( \alpha_0 \) in the plane triangle with the same sides, the following estimates hold:

\[ \alpha - \alpha_0 \leq \omega^+(G), \]

\[ \alpha_0 - \alpha \leq 2\omega^+(G) + \omega^-(G) \leq 2\Omega(G). \]

If \( T \) is a reduced convex triangle, then

\[ \alpha - \alpha_0 \leq \omega^+(\bar{T}), \]

\[ \alpha_0 - \alpha \leq 2\omega^+(\bar{T}) + \omega^-(\bar{T}) \leq 2\Omega(\bar{T}), \]

where \( \bar{T} \) is the set of all points of the triangle \( T \), including its sides and vertices.

Proof. Inequality (71) follows from Theorem 5 of Chapter II. In fact, suppose that \( t_A \) are the triangles cut off by shortest arcs lying in \( G \) which join points on the sides of the angle \( \alpha \). The triangles \( t_A \) may be supposed to be reduced. We will denote the upper angle by a superscript bar. Then we have:

\[ \alpha - \alpha_0 = \bar{\alpha} - \alpha_0 \leq \nu_\lambda \leq \sup \bar{\delta}^+(t_A) = \sup \delta^+(t_A) \leq \omega^+(G). \]

In this chain of relations the equality follows from the existence of the
angle, the first inequality from Theorem 5 of Chapter II, the next from the definition of $\nu^+$ [formula (20) of Chapter II], and the last from the definition of $\omega^+(G)$.

Inequality (72) is obtained as follows:

$$(\alpha - \alpha_0) + (\xi - \xi_0) + (\eta - \eta_0) = \delta(T_0),$$

so that

$$\alpha_0 - \alpha = (\xi - \xi_0) + (\eta - \eta_0) - \delta(T) \leq \omega^+(G) + \omega^+(G) + \omega^-(G).$$

In the case when the reduced triangle is moreover convex, the region $G$ may be chosen arbitrarily and enclosing $T$, so that inequalities (71) and (72) go into inequalities (73), (74).
1. Direction and rotation of curves.

1. Direction of a curve. The concept of direction of a curve was introduced in subsection 2 of Chapter II. The curve $L$ has at its initial point a definite direction if it forms with itself a definite angle at that point. If that angle exists, its value can be equal only to zero.

In Chapter II we were dealing with curves in arbitrary metric spaces. We turn now to curves in two-dimensional manifolds of bounded curvature.

Lemma 1. Suppose that the curves $L$ and $M$ issue from the point $0$ and $X \in L$, $Y \in M$. In order that there should exist an angle $\alpha(L, M)$ at the point $0$ between these curves, it is necessary and sufficient that the limit $\lim_{X, Y \to 0} \alpha(OX, OY)$ should exist, where $\alpha(OX, OY)$ is the angle between the shortest arcs $OX$ and $OY$ (any such shortest arcs if there are several). In addition

$$\alpha(L, M) = \lim_{X, Y \to 0} \alpha(OX, OY).$$

The approach of $X$ to $0$ and $Y$ to $0$ is understood in the sense of parameters on $L$ and $M$.

Proof. From Theorem 4 of Chapter IV, the angle $\alpha(OX, OY)$ exists. Suppose that $\gamma(X, Y)$ is the angle corresponding to it in the plane triangle with sides $OX$, $OY$, $XY$. From Theorem 11 of Chapter IV, for any $\varepsilon > 0$

$$|\alpha(OX, OY) - \gamma(X, Y)| < \varepsilon$$
given only that the points $X$ and $Y$ are sufficiently close to $O$. Therefore from the existence of $\lim_{X, Y \to 0} \alpha(OX, OY)$ follows the existence of the limit $\lim_{X, Y \to 0} \gamma(XY)$ equal to it and conversely. But this means that (1) is satisfied.

Theorem 1. If the curves $L = X(t)$ and $M = Y(s)$ ($0 \leq t, s \leq 1$) issuing from $O$ have definite directions, then they form at $O$ a definite angle. If each of the curves $L, M$, issuing from the point $0$, on some initial segment no longer passes through $O$ and if at the point $O$ the curves $L$ and $M$ form a definite angle, less than $\pi$, then each of the curves $L$ and $M$ has at $O$ a definite direction.
Proof. 1. Denote by $\alpha(t,s)$ the angle between the shortest arcs $OX(t)$, $OY(s)$. The function $\alpha(t,s)$ may be multiple-valued, if there are several shortest arcs $OX(t)$ and $OY(s)$. Analogously we denote by $\xi(t,t')$ and $\eta(s,s')$ the angles between the shortest arcs $OX(t)$, $OX(t')$ and between the shortest arcs $OY(s)$, $OY(s')$ respectively. If the curves $L, M$ have directions at $O$, then from Lemma 1

$$\lim_{t, t' \to 0} \xi(t, t') = 0, \quad \lim_{s, s' \to 0} \eta(s, s') = 0.$$ 

By the fundamental theorem on upper angles

$$\alpha(t, s) \leq \xi(t, t') + \alpha(t', s') + \eta(s, s'),$$

$$\alpha(t', s') \leq \xi(t, t') + \alpha(t, s) + \eta(s, s').$$

Therefore from the smallness of $\xi$ and $\eta$ for sufficiently small $t, t', s, s'$ it follows that $|\alpha(t, s) - \alpha(t', s')|$ is small and thus that $\lim_{t, s \to 0} \alpha(t, s)$ exists. By Lemma 1, this is sufficient for the existence of the angle between $L$ and $M$. The first assertion of Theorem 1 is proved.

2. Suppose that at $O$ there exists an angle between $L$ and $M$ and that this angle $\alpha < \pi$. Then from Lemma 1 the limit

$$\lim_{t, s \to 0} \alpha(t, s) = \alpha < \pi$$

exists. We choose an arbitrarily small number $\varepsilon > 0$, $\varepsilon < \pi - \alpha$ and choose a $\delta > 0$ such that for any $0 < t, s, t', s' < \delta$ the inequality

$$|\alpha(t, s) - \alpha(t', s')| < \varepsilon$$

holds and also that for $t, s < \delta$ the points $X(t), Y(s)$ do not leave a neighborhood of the point $O$ within the limits of which the angle between the shortest arcs $OA, OB$ differs from the corresponding angle $\gamma(A, B)$ in the plane triangle by less than $\varepsilon$. Inequality (3) is guaranteed because of the existence of the limit (2), and the supplementary condition because of Theorem 11 of Chapter IV. Passing in inequality (3) to the limit as $t', s' \to 0$, we have also for $0 < t, s < \delta$

$$|\alpha(t, s) - \alpha| \leq \varepsilon.$$ 

We assert that for $0 < t, s, t', s' < \delta$ we will have the estimates

$$\xi(t, t') < 3\varepsilon, \quad \eta(s, s') < 3\varepsilon.$$ 

For definiteness we shall prove the first of inequalities (5).

3. Suppose that, contrary to what we are trying to prove, there exist $t_1, t_2$ and $OX(t_1), OX(t_2)$ such that $0 < t_1 \leq t_2 < \delta$ and

$$\xi(t_1, t_2) \geq 3\varepsilon.$$
The shortest arcs $OX(t_1)$, $OX(t_2)$ cannot have common points besides $O$, for otherwise for certain points $A, B$ on them we would have $\gamma(A, B) = 0$. Then by the choice of $\delta$ the angle between these shortest arcs would be less than $\varepsilon$, which contradicts inequality (6). Therefore the shortest arcs $OX(t_1)$, $OX(t_2)$ divide the neighborhood of the point $O$ into two sectors. By the same principle $t_1 \equiv t_2$.

Choose $0 < s < \delta$ and draw the shortest arc $OY(s)$ forming definite sectors with $OX(t_i)$ and $OX(t_2)$. One of these sectors may be degenerate, having coincident sides close to the vertex. From inequality (4) and the condition $\varepsilon < \pi - \alpha$ it follows that the angles formed by $OY(s)$ with $OX(t_i)$ and $OX(t_2)$ are less than $\pi$. Suppose that $U_1, U_2$ are those of the sectors bounded by the shortest arcs $OY(s)$, $OX(t_i)$ and $OX(t_2)$, whose angles are equal to the angles between these shortest arcs. Such sectors exist, since these angles are less than $\pi$ (see Theorem 6 of Chapter IV). The sector angles of $U_1, U_2$, equal to the angles between their sides, differ from one another, in view of inequality (3), by less than $\varepsilon$. Therefore the sectors $U_1, U_2$ lie on different sides of $OY(s)$, as shown in Figure 72, for otherwise the shortest arcs $OX(t_1)$, $OX(t_2)$ would form an angle less than $\varepsilon$ in contradiction with (6).

4. Choose any $t$ between $t_1$ and $t_2$ and draw a shortest arc $OX(t)$, forming, with those earlier laid off, definite sectors. One of the sectors formed by $OX(t)$ with $OY(s)$ has an angle equal to the angle $\alpha(t,s)$ between its sides. We denote this sector by $U$. If $U$ lies on the same side of $OY(s)$ as $U_1$, then $\xi(t, t_1) < \varepsilon$. If $U$ lies on the same side of $OY(s)$ as $U_2$, then $\xi(t, t_2) < \varepsilon$.

5. Suppose that $\rho$ is the distance from $O$ to the closed arc $[X(t_1), X(t_2)]$ of the curve $L$. By the hypotheses of the theorem the curve $L$ issuing from $O$ does not again intersect $O$. Therefore $\rho > 0$.

Above we verified the fact that for $t_1 < t < t_2$ there exists a shortest arc $OX(t)$ forming a angle less than $\varepsilon$ with either $OX(t_1)$ or $OX(t_2)$. If $t = t_1$ we may take as such a shortest arc the original shortest arc $OX(t_1)$, which forms with itself a null angle, and if $t = t_2$ a shortest arc forming a null angle with $OX(t_2)$. Dividing $[X(t_1), X(t_2)]$ into fine pieces, we discover on $L$ points $X(t'), X(t'')$ very close to one another even in comparison with
ρ and such that there exists a shortest arc $OX(t')$ forming an angle less than $ε$ with $OX(t_1)$ and a shortest arc $OX(t'')$ forming an angle less than $ε$ with $OX(t_2)$.

Because $X(t')$ and $X(t'')$ are very close in comparison with $ρ$, the angle $\gamma[X(t'), X(t'')]$ will be arbitrarily small and we may suppose that $\xi(t', t'') < ε$. Finally we have

$$\xi(t_1, t_2) \leq \xi(t_1, t') + \xi(t', t'') + \xi(t'', t_2) < 3ε,$$

which contradicts proposition (6).

Theorem 1 is completely proved.

Remark. The second assertion of Theorem 1, generally speaking, is no longer valid if $α(L, M) ≥ π$, and also if one of the curves can pass several times through the point $O$. A first example is given in Remark 7 of subsection 2 of Chapter II. A second is easily constructed on the plane. Suppose that $L$ is a ray with origin $O$ and $M$ is a curve which comes into the point $O$, making infinitely many ever finer oscillations from $O$, now along the ray $M'$, and now along the ray $M''$, and suppose that the rays $M'$ and $M''$ form with $L$ equal angles $α$ and lie one to the left and one to the right of $L$.

2. Side of a curve. Suppose that for the simple arc $L$ with endpoints $A$ and $B$ one of the three following constructions has been carried out: 1) the arc $L$ is surrounded by a region $G_1$ homeomorphic to the disc, and the endpoints $A$ and $B$ of the arc $L$ are joined by simple curves $l_A, l_B$ with points on the boundary of the region $G_1$ such that $l_A + L + l_B$ form one simple curve (Figure 73a); 2) the arc $L$ is surrounded by a region $G_2$ homeomorphic to the disc, and the endpoints $A$ and $B$ of the arc $L$ are joined by a simple curve $L'$ lying in $G_2$ having no common points with $L$ other than the endpoints (Figure 73b); 3) an arc $A_0B_0$, obtained by rejecting some endpieces of the arc $L$, is surround by a region $G_3$ homeomorphic to the disc, with the endpoints $A$ and $B$ of the arc $L$ lying outside $G_3$ (Figure 73c).
Any of these three constructions makes it possible to assign a precise sense to the statement that certain points of the corresponding region $G$ lie on one or on another side of $L$. In the first case the curve $l_A + L + l_B$ divides the points of the set $G_1 - (l_A + L + l_B)$ into two classes. In the second case the closed curve $L + L'$ divides the points of the set $G_2 - (L + L')$ into two classes. In the third case the piece $A_iB_i$ of the curve $L$ divides the points of the set $G_3 - A_iB_i$ into two classes. Here $A_iB_i$ is the piece from the last point of the intersection of $L$ with the boundary of $G_3$ preceding $A_0$ to the first points of the intersection of $L$ with the boundary of $G_3$ following $B_0$. We shall not dwell on the proof of the possibility of realizing any of the constructions of Figure 73 in many ways for each simple arc $L$ in a two-dimensional manifold. In the required cases the corresponding construction will be carried out.

For each of the three definitions given above, the region of points for which the property of lying on a definite side of $L$ is defined and the division of them into two classes, depends on the choice of $G_1, l_A, l_B; G_2, L'; G_3, A_0, B_0$. The situation is different if we turn to some limiting relations.

Suppose on the curve $L$ there is fixed some interior point $M$ and that there is a sequence of points $M_n$ converging to $M$ and not lying on $L$. We shall say that the sequence $M_n$ converges to $M$ from a definite side of $L$ if all the points $M_n$ from a certain point onward lie on a definite side of $L$ in the sense of any of the definitions 1), 2), 3). In the last case we suppose that the point $M$ is contained in the piece $A_0B_0$ of the curve $L$.

The following state of affairs is easily verified. The property of the sequence $M_n$ of converging to an interior point $M$ of the curve $L$ from a definite side of $L$ does not depend on the choice of the first, second, or third of the definitions given above, on the choice of $G_1, l_A, l_B$ in the first definition, on the choice of $G_2, L'$ in the second definition, or on the choice of the piece $A_0B_0$ containing $M$ and the region $G_3$ in the third definition.

Now we suppose that the point $M$ inside $L$ and a sequence of points $M_n$ converging to $M$ from a definite side of $L$ are fixed. For each construction of the type of Figure 73 (in the case 73c we suppose that $M \in A_0B_0$), the two classes of points lying on the different sides of $L$ divide into a class containing an infinite part of the sequence $M_n$ and a class not containing an infinite part of that sequence. The classes formed by one of these tests (it makes no difference to us which) will be called
classes lying to the left, and the other classes lying to the right of $L$. This division does not depend on the concrete choice of $M$ and $M_n \rightarrow M$. In this sense we may speak of points lying to the right or left of $L$. If the two-dimensional manifold in question is a surface in three-dimensional Euclidean space and the direction of passing along $L$ is fixed, and the choice of a side of the surface is extended by continuity along the curve $L$ and with $L$ extended to the entire region $G_1, G_2, G_3$, then the discrimination of points as lying to the right or to the left of $L$ takes on a quite intuitive sense under the conditions of Figure 73.

Now suppose that a sequence of simple curves $L_n$ converges to the simple curve $L$ in the metric $\rho$ of the manifold in question. Suppose that each of the $L_n$ either has no points in common with $L$ or has no common points with $L$ other than one or both common endpoints. We shall say that the curves $L_n$ converge to $L$ from a definite side if for some choice of the parameter $0 \leq t \leq 1$ on the curves $L_n = X_n(t), L = X(t)$, not only is $\lim_{n \to \infty} \rho(X_n(t), X(t)) = 0$ uniformly in $t$ but also for each $0 < t_0 < 1$ the points $X_n(t_0)$ converge to $X(t_0)$ from a definite side of $L$. In this case all the points $X_n(t)$ converge to the corresponding points $X(t)$ from the right, or they all converge from the left. Hence we may correspondingly say that $L_n$ converges to $L$ from the right or from the left.

The following sufficient conditions are true in a trivial way. 1) If for the curve $L$ we carry out a construction as in Figure 73a, and all interior points of the curves $L_n \rightarrow L$ lie in one of the classes into which $I_A + L + I_B$ divides $G_1 - (I_A + L + I_B)$, then $L_n$ converges to $L$ on the corresponding side. 2) If for $L$ we carry out a construction of type Figure 73b, and all the $L_n \rightarrow L$ with the exception possibly of their endpoints, and they lie in $G_2$ on one side of $L + L'$, then the $L_n$ converge to $L$ on the corresponding side. We note further without proof a condition that is not only sufficient but necessary. If curves $L_n$, having no points in common with $L$ except possibly common endpoints, converge to $L$, i.e., in some parametrization $0 \leq t \leq 1$, $L_n = X_n(t), L = X(t)$ the convergence $X_n(t) \rightarrow X(t)$ is realized uniformly, and if for at least one $t_0$ with $0 < t_0 < 1$ the points $X_n(t_0)$ converge to $X(t_0)$ on a definite side of $L$, then also for all $0 < t < 1$ the convergence $X_n(t) \rightarrow X(t)$ is realized on a definite side, in fact on the same side, of $L$.

Finally, we note that if the curve $L$ is enclosed in a region $G$ homeomorphic to the disc and the endpoints of $L$ are joined by a curve $L_n$ lying in $G$ and having no common points with $L$ other than endpoints, then, fixing the orientation of $G$ and the direction of running through
the curve $L$, we may say that $L_n$ passes to the left or to the right of $L$
depending on whether the orientation of the contour $L + L_n$ coincides with
the orientation of $G$ or is opposite to it. Evidently, if the sequence of
curves $L_n$ of the indicated type converges to $L$ and all of them pass on
one side of $L$ in the sense of the last definition, then they converge to
$L$ on a definite side in the sense of the preceding definition.

3. Rotation of a curve. Suppose that $L$ is a simple curve with a
definite direction at each of its endpoints $A$ and $B$, and let $L_n$ be an
arbitrary sequence of broken lines joining the endpoints of the curve $L$,
having no other points in common with $L$, and converging to $L$ on
the right. If at the endpoints $A, B$ the complete angle $\theta \neq 0$, then such broken
curves $L_n$ certainly exist. Indeed, $L$ may be encircled by a region $G$
homeomorphic to the disc. One may pass shortest arcs $AA'_n$ and $BB'_n$
not intersecting $L$ from the endpoints $A$ and $B$. One may then encircle $L$ in
$G$ by a simple closed broken curve $L'_n$ very close to $L$. Suppose that $A_n$
and $B_n$ are the closest to $A$ and $B$ of the intersection points of $L'_n$ with
$AA'_n$ and $BB'_n$. Choosing one of the pieces $\widehat{A_nB_n}$ of the broken curve $L'_n$,
we form $L_n = AA_n + A_nB_n + B_nB$. These broken curves converge to $L$ on
one side.

Suppose that $\bar{\alpha}_n$ and $\bar{\beta}_n$ are sector angles between $L$ and $L_n$ at the points
$A$ and $B$ from the side of the region encircled in $G$ by the contour $L + L_n$,
and suppose that $\bar{\varphi}_{ni}$ are the sector angles at the remaining vertices of $L_n$
from the outer side of the contour $L + L_n$.

The right rotation of the curve $L$, more precisely the right rotation
of the open arc $AB$ of the curve $L$, is the limit

$$
\tau_r(L) = \lim_{n \to \infty} \left[ \sum_i (\pi - \bar{\varphi}_{ni}) + \bar{\alpha}_n + \bar{\beta}_n \right].
$$

The existence of this limit is proved below. Analogously, using broken
curves $L_n$ converging to $L$ from the left, one defines the left rotation $\tau_l(L)$.

Remarks. 1) We could have restricted ourselves to broken curves for
which the angles $\alpha_n, \beta_n \to 0$. Then

$$
\tau_r(L) = \lim_{n \to \infty} \sum_i (\pi - \varphi_{ni}).
$$

2) Among the angles $\varphi_{ni}$ we may, along with the vertices of the broken
curves $L_n$, include all the points $A_{ni}$ on the sides of $L_n$ at which the
corresponding angles $\varphi_{ni} \leq \pi$.

There may be infinitely many such points, but for them
so that they cannot form more than a countable set. By the same principle \( \sum_{j} (\pi - \phi_{nj}) \to 0 \) as \( n \to \infty \), since the interior points of \( L_n \) are contained in a vanishing sequence of sets \( (G_n - L) \), and therefore \( \lim_{n \to \infty} \sum_{j} \omega(A_{nj}) = 0 \).

3) Definition (7) may be replaced by the following:

\[
\tau_r(L) = \lim_{n \to \infty} \left[ \sum (\phi_{ni} - \pi) + \alpha_n + \beta_n \right],
\]

where the \( \phi_{ni} \) are sector angles at the vertices of \( L_n \) from within the contour \( L + L_n \). Indeed, the difference of the quantities in brackets in equations (7) and (8) is equal to \( \sum (\phi_i + \phi_i - 2\pi) \), i.e., the sum of the curvature at nonendpoint vertices of \( L_n \), and this last tends to zero as \( n \) increases.

**Theorem 2.** A curve without multiple points with definite directions at the endpoints (under the hypothesis that the endpoints lie at points with complete angle \( \theta \neq 0 \)) always has a definite rotation, i.e. for such a curve the limit (7) exists.

**Proof.** Enclose \( L \) in a region \( G \) with absolute curvature \( \Omega(G - L) < \epsilon \). Suppose that \( L_m \) and \( L_n \) are broken curves lying in \( G \) from the sequence figuring in the definition (7). Consider the difference

\[
q_{mn} = \left[ \sum (\pi - \phi_{ni}) + \alpha_n + \beta_n \right] - \left[ \sum (\pi - \phi_{mk}) + \alpha_m + \beta_n \right].
\]

1. We first suppose that \( L_m \) and \( L_n \) have no common points other than common endpoints or common end-segments. For definiteness we suppose that \( L_m \) lies to the right of \( L_n \) as in Figure 74. Then, with the notations indicated in Figure 74, we will have

\[
q_{mn} = d(P) - \sum \omega(A_{mk}).
\]

If portions of \( L_m \) or \( L_n \) coincide near the vertices \( A \) or \( B \), then the polygon \( P \) cannot be adjacent to the corresponding vertex. But if \( P \) is adjacent to the vertex \( A \) or \( B \), it can by Lemma 22 of Chapter V be cut near that vertex, resulting in a polygon \( P \) with an excess differing by less than \( \epsilon \) from the excess \( d(P) \). Therefore
\[ |q_{mn}| < |\hat{d}(P')| + \sum_k |\omega(A_{mk})| + \varepsilon < \Omega(G - L) + \varepsilon. \]

2. Now we suppose that \( L_m \) and \( L_n \) have common points other than the endpoints or adjacent to the endpoints of common segments. In this case we may pass a third broken curve \( L_p \), going to the right of \( L_m \) and \( L_n \) and having only common endpoints or common end-segments with \( L_m \) and \( L_n \). Then

\[
|q_{mn}| \leq |q_{mp}| + |q_{np}| < 2\Omega(G - L) + 2\varepsilon.
\]

Thus, for \( L_m, L_n \) lying in a region \( G \) corresponding to \( \varepsilon \), we always have \( |q_{mn}| < 4\varepsilon \). Because of the arbitrary smallness of \( \varepsilon > 0 \) this shows that the limit (7) exists. Theorem 2 is proved.

We note that, putting \( m \to \infty, \varepsilon \to 0 \) in (11), we get

\[
|\tau_r(L) - \left[ \sum \phi_n + \alpha_n + \beta_n \right]| \leq 2\Omega(G - L_n).
\]

Remark. If at least one of the endpoints of the curve \( L \) lies at a point where \( \theta = 0 \), then the curve may not have a definite rotation even in the presence of definite directions at its endpoints. It is easy to construct an appropriate example, selecting a curve of the form of the curve \( L \) in Figure 75.

It is clear from the proof of Theorem 2 that if from an endpoint of \( L \) where \( \theta = 0 \) one can pass a shortest arc having no common points other than the origin with \( L \), then the rotation of \( L \) will exist.

2. Connection between rotation and curvature.

4. The sum of the left and right rotations.

Theorem 3. The sum of the rotations of the simple curve \( L \) with endpoints \( A, B \) is equal to the curvature of that curve as a set with the endpoints of the curve deleted:

\[
\tau_r(L) + \tau_l(L) = \omega(L - A - B).
\]

Proof. We are of course assuming that the endpoints \( A, B \) lie at points with complete angles \( \theta = 0 \), and that \( L \) has definite directions at the endpoints. Marking off from \( A \) and \( B \) along a pair of shortest arcs forming nonnull angles with \( L \), and joining their endpoints, we enclose \( L \) in a closed polygonal curve \( L_0 \) which bounds a region \( G \) homeomorphic to the
open disc (Figure 76). The curve \( L \), with the exclusion of its endpoints, lies in \( G \) and divides \( G \) into two parts. The region \( G \) may be chosen so that \( \Omega(G - L) < \varepsilon \), so that

\[
|\omega(G) - \omega(L - A - B)| < \varepsilon.
\]

In the region \( G \), by Theorem 9 of Chapter V, there exists a polygon \( Q \) such that for any \( P' \subset G \) and containing \( Q \)

\[
|\delta(P') - \omega(G)| < \varepsilon.
\]

We join the points \( A \) and \( B \) in \( G \) by broken curves \( L_1 \) and \( L_2 \) going to the right and left of \( L \) and containing \( Q \). In view of (12) we will have

\[
|\tau_r(L) + \tau_l(L) - \left[ \sum (\pi - \phi_i) + \alpha + \beta \right]| \leq 4\Omega(G - L) < 4\varepsilon.
\]

Moreover, for the polygon \( P \) bounded by the broken lines \( L_1, L_2 \)

\[
\left| \left[ \sum (\pi - \phi_i) + \alpha + \beta \right] - \delta(P) \right| \leq \left| \sum \omega(M_i) \right| \leq \Omega(G - L) < \varepsilon.
\]

By Lemma 22 of Chapter V, the sectors \( P \) adjacent to \( A \) and \( B \) may be cut, replacing \( P \) by a polygon \( P' \) which also contains \( Q \) but already entirely lies in \( G \), while satisfying the inequality

\[
|\delta(P') - \delta(P)| < \varepsilon.
\]

The validity of inequality (15) follows for \( P' \).

From inequalities (14)—(18) follows

\[
|\tau_r(L) + \tau_l(L) - \omega(L - A - B)| < 8\varepsilon,
\]

which, because of the arbitrariness of \( \varepsilon > 0 \), proves equation (13).

5. Additivity of the rotation. Suppose on the curve \( L \) we mark off a point \( G \) inside a simple arc \( A, B \) having rotation, and that both branches of the curve \( L \) have a definite direction at \( C \). The right rotation of the curve \( L \) at the point \( C \) will be the quantity

\[
\tau_r(C) = \pi - \bar{\alpha},
\]

where \( \bar{\alpha} \) is the sector angle, which at the point \( C \) is bounded by the branches of the curve \( L \) to the right of \( L \).

Theorem 4. If under the indicated conditions the complete angle around
the point C is different from zero, then

\[ (19) \quad \tau(\overline{AB}) = \tau(\overline{AC}) + \tau(\overline{C}) + \tau(\overline{CB}) \]

for the left and right rotations.

**Proof.** Because of Theorem 3 it suffices to prove (19) for the rotations on one of the sides. We shall consider that side for which \( \bar{\alpha} = 0 \). In this case, using the rotations indicated in Figure 77, we may carry out the construction depicted on Figure 77 in such a way that the broken curves \( L_0, L_1, L_2 \) bound a polygon homeomorphic to the disc and that the inequalities

\[
\left\{ \begin{array}{l}
|\tau(\overline{AB}) - \left[ \sum \phi_i - \pi \right] + \alpha_0 + \beta_0 | < \varepsilon, \\
|\tau(\overline{AC}) - \left[ \sum (\pi - \phi_{ij}) \right] + \alpha_1 + \beta_1 | < \varepsilon, \\
|\tau(\overline{CB}) - \left[ \sum (\pi - \phi_{2k}) \right] + \alpha_2 + \beta_2 | < \varepsilon,
\end{array} \right.
\]

are satisfied. Noting that

\[
\delta(P) = \sum \phi_i - \sum (\pi - \phi_{ij}) - \sum (\pi - \phi_{2k}) + (\beta_0 - \beta_2 - \pi) + (\alpha - \beta_1 - \alpha_2 - \pi) + (\alpha_0 - \alpha_1 - \pi) + 2\pi,
\]

we conclude from inequalities (20) that the difference of the right and left sides of (19) is smaller than 5\( \varepsilon \) in absolute value. Because of the arbitrariness of \( \varepsilon > 0 \), (19) follows.

6. Possible peculiarities. 1) In Theorem 2 we required not only that the curve have definite directions at the endpoints, but also that the complete angles around the endpoints should be different from zero. The necessity for this requirement was affirmed by example.

Analogously, in Theorem 4 the requirement \( \theta(C) \neq 0 \) is essential. If \( \theta(C) = 0 \) it may generally speaking turn out that the rotations of the arcs \( AC \) and \( CB \) cannot be separately assigned finite values.

2) The concepts of a side of the curve \( L \), the convergence of a sequence
of curves $L_n$ to $L$, of a definite side of $L$, and the concept of the rotation of the curve $L$ were all defined for curves $L$ not having multiple points. However, these concepts may be extended also to the case of curve $L$ having a finite number of interior points of finite multiplicity. In this case we may enclose $L$ in a neighborhood $G$ and construct another metrized region $G'$ homeomorphic to the disc and locally isometric to $G'$, such that $G$ may be considered a locally isometric covering over $G'$, overlapping itself near the multiple points of $L$, as depicted in Figure 78. In the abstract covering region $G$, the curve $L$ does not have multiple points, and the left and right sides are defined for it in $G$ along with the convergence of curves $L_n$ to $L$ from a definite side and the left and right rotations of $L$.

For such a construction Theorem 2 on the existence of rotation turns out to be valid, along with Theorem 3 to the effect that the sum of the right and left rotations is equal to the curvature of the curve, the curvature at multiple points being taken with the corresponding multiplicity. Also Theorem 4 on the additivity of rotation is valid. Therefore if a curve with multiple points is decomposed into a finite number of simple pieces, and if at the points of the subdivision the complete angle $\theta \neq 0$, and the branches of the curve have directions, then by the rotation of the curve we may understand simply the sum of the rotations of the pieces and the rotations at the points of the subdivision.

3) In the case of a closed curve, even one which has no multiple points, the concepts of left and right sides of the curve lose their meaning. As an example of this we may note the centerline curve on the Möbius strip. But if a closed curve in a two-dimensional manifold is the boundary of a region, then we may certainly distinguish two sides of the curve, the interior and exterior, relative to this region.

Suppose that a simple closed curve $L$ bounds a region $G$ homeomorphic to the disc. Consider a sequence of simple closed polygonal curves $L_n$ converging to $L$ from one side of $G$; the sector angles from the side of $L$ at the vertices of $L_n$ are denoted by $\phi_n$. The rotation $\tau_i(L)$ of the curve $L$ on the interior side of $G$ will be

$$\tau_i(L) = \lim_{n \to \infty} \sum_i (\pi - \phi_n).$$

Analogously we define the rotation $\tau_e(L)$ of the curve $L$ on the exterior side of $G$. 

![Figure 78.](image-url)
As in Theorem 2 we prove that the limit (21) exists. This time it is not necessary to require the existence on $L$ of points with definite directions. As in Theorem 3, we prove that $\tau_i(L) + \tau_c(L) = \omega(L)$. Finally, if there is on $L$ a point $C$ with a complete angle $\theta \neq 0$ and if at this point both branches of $L$ have directions, then the rotation of $L$ decomposes into the rotation of the open arc from $C$ to $C$ and the rotation at the point $C$ itself. The proof is quite analogous to that of Theorem 4.

7. The Gauss-Bonnet theorem. From definition (21) and from the definition of curvature in terms of polygons (Theorem 9 of Chapter V), we have the following extension of the Gauss-Bonnet theorem to the case of two-dimensional manifolds.

**Theorem 5.** If the simple closed curve $L$ bounds a region $G$ homeomorphic to the disc, then

$$\omega(G) = 2\pi - \tau_i(L).$$

If several simple closed curves $L_n$ make up the boundary of a region $G$ with compact closure and the Euler characteristic is $\chi(G)$, then formula (22) is replaced by

$$\omega(G) = 2\pi\chi(G) - \sum_n \tau_i(L_n).$$

3. Rotation of shortest arcs.

8. One-sided approximation of a shortest arc.

**Lemma 2.** For every shortest arc $L$ we may construct a sequence of simple polygonal curves $L_n$ converging to $L$ from a given side, each of which has no points in common with $L$, and such that the lengths of the $L_n$ converge to the length of $L$.

Later on, in Chapter IX, we shall investigate curves with rotations having bounded variation. There we shall see that that the required sequence of polygonal curves $L_n$ may be constructed quite arbitrarily. Here, in the construction of $L_n$, we use a more special construction.

1. Suppose that $A$ and $B$ are the endpoints of the shortest arc $L$. We may carry out the proof for a curve $L$ for which the absolute curvature $\Omega(L - A - B)$ is small, say

$$\Omega(L - A - B) < \frac{1}{10}.$$

Indeed, every other shortest arc $L$ may be decomposed by points $C_i$
into a finite number of pieces of the above sort. Enclosing each point \( C_i \) in a closed polygon in the form of a loop \( l_i \) of very small length and approximating the shortest arc \( C_iC_{i+1} \) on each piece by a polygon \( L_{ni} \), close in length to \( C_iC_{i+1} \), we may then form from the pieces of the polygon \( L_{ni} \) and the pieces of loops \( l_i \) the required simple polygon \( L_n \). If the loops \( l_i \) close down on the \( C_i \) and become infinitely short, and the \( L_{ni} \) and their lengths converge to \( C_iC_{i+1} \), then the polygonal curves obtained in the above way will satisfy the conditions of Lemma 2.

From what has been said, we may from now on suppose that (24) is satisfied.

2. We enclose \( L \) in a neighborhood \( G \) homeomorphic to the disc, in which all the necessary constructions will be carried out. We may choose \( G \) so that the inequalities

\[
\Omega(G - L) < \varepsilon_1, \quad \Omega(G - A - B) < \frac{1}{10},
\]

are satisfied. Then we choose points \( A_0, B_0 \) on \( L \) such that the distances \( AA_0 \) and \( B_0B \) will be small in comparison with the distance \( \rho' \) from \( A \) to \( B \):

\[
AA_0 = B_0B = \rho'\varepsilon_2.
\]

The segment \( A_0B_0 \) of the shortest arc \( L \) will now be enclosed in a polygon \( P \) lying in \( G \) and homeomorphic to the disc, such that that segment lies inside \( P \) and the points \( A \) and \( B \) lie outside \( P \) (Figure 79). Suppose that \( \rho'' \) is the smallest distance from \( A_0B_0 \) to the boundary of \( P \) and \( \rho''' \) the smallest distance from the boundary of \( P \) to the boundary of \( G \).

We introduce a quantity \( \varepsilon_3 \), very small in comparison to \( \rho' \), \( \rho'' \) and \( \rho''' \), and a quantity \( \varepsilon_4 \) very small in comparison with \( \varepsilon_3 \). We triangulate the polygon \( P \) into triangles homeomorphic to the disc with diameters less than \( \varepsilon_4 \). Then we reject those of the triangles of this triangulation for which at least one point lies at a distance not larger than \( \varepsilon_3 \) from \( A_0B_0 \). The remaining triangles of the collection include in particular all

![Figure 79.](image)
triangles adjacent to the boundary of $P$, since $\varepsilon_3 + \varepsilon_4 \ll \rho''$, so that the reduced collection is not empty. It may now be decomposed into two components. The triangles adjacent to the boundary of $P$ will belong to one connected component. The interior boundary $L_o$ of this connected component will be a simple closed polygon. The broken curve $L_o$ will enclose the segment $\overline{A_0B_0}$ and it will pass at an “almost constant” distance from $\overline{A_0B_0}$: for each point $X \in L_o$

$$\varepsilon_3 \leq \rho(X, \overline{A_0B_0}) \leq \varepsilon_3 + \varepsilon_4.$$  

Suppose that $A_1B_1$ is the piece of $L$ from the last intersection of $L$ with the boundary $P$ preceding $A_0$ up to the first intersection of $L$ with the boundary of $P$ following $B_0$. The segment $A_1B_1$ divides $P$ into two parts $P_1, P_2$, as in Figure 79.

The polygon $L_o$ can intersect $L$ only inside the piece $\overline{A_1B_1}$, since $\varepsilon_3 \ll \rho'''$. There may be many intersections of $L_o$ with $L$, but they will be grouped in two regions, close to $A_0$ and close to $B_0$, since $\varepsilon_3 \ll \rho'$. Suppose on a circuit in a definite direction around the contour $L_0$ the points $B_2$ and $B_3$ are the first and last intersections of $L_0$ with $L$ close to $B_0$, and $A_3, A_2$ the first and last intersections of $L_0$ with $L$ near to $A_0$ (Figure 79). In the semineighborhood $P_1$ which the segment $\overline{A_1B_1}$ excises from $P$ the piece $\overline{A_2B_2}$ of the polygonal curve $P_0$ is a simple arc with endpoints on $L$.

3. Each point $X \in \overline{A_2B_2}$ may be joined by one or several shortest arcs $XM$ to one or several points $M \in \overline{A_0B_0}$ closest to $X$. The lengths of the shortest arcs $XM$ lie within the limits $\varepsilon_3, \varepsilon_3 + \varepsilon_4$, because of (27).

All the shortest arcs $XM$ pass inside $P$. We are going to consider only those of them which lie in $P_1$. If some shortest arc $XM$ leaves $P_1$, then it, not enjoying the possibility of intersecting the boundary of $P$ ($\varepsilon_3 \ll \rho'''$), intersects $\overline{A_1B_1}$. But then the piece of the shortest arc $XM$ issuing from $P$ may be the corresponding segment of the shortest arc $\overline{A_1B_1}$.

The shortest arc $XM$ lies in $P_1$ but does not necessarily lie in the region $\widetilde{G}$ bounded by the arc $\overline{A_2B_2}$, the polygon $L_o$ and the segment $\overline{A_2B_2}$ of the shortest arc $L$.

Suppose that several points $X_i \in \overline{A_2B_2}$ are joined by shortest arcs $X_iN_i$ with the segment

![Figure 80](https://via.placeholder.com/150)
\( A_0B_0 \), with the shortest arcs \( X_t M_t \) issuing from \( G_0 \), as in Figure 80. Replacing the corresponding parts of the arc \( L_0 = A_2B_2 \) by the pieces of these shortest arcs, we obtain instead of \( A_2B_2 \) a new polygon \( L' \), the points \( X \) of which are, as before, distant from \( A_0B_0 \) by distances lying within the limits (27). At the same time \( L' \) encloses a region \( G' \) wider than \( G_0 \).

Suppose that \( G_0'' \) is the sum of all the regions \( G' \) which may be obtained in the indicated way for all possible choices of points \( X_t \) and shortest arcs from \( X_t \) to \( A_0B_0 \). The interior of \( P_1 \), in which all the \( G' \) lie, is homeomorphic to the disc and has a countable basis. Therefore there exist curves \( L_n' \) constructed analogously to \( L' \), for which the regions \( G_n' \) will, successively increasing with the growth of \( n \), exhaust in their sum \( G_0'' \).

The curves \( L_n' \) lie in a compact region \( P_1 \). Their lengths are uniformly bounded, since they are not longer than \( A_2B_2 \) and do not grow with the growth of \( n \). Therefore, by dropping some of the \( n \) we may suppose that the \( L_n' \) converge to some limit curve \( \bar{L} \). The curve \( \bar{L} \) will bound \( G_1'' \). Moreover \( \bar{L} \) is a simple arc. Indeed, it is the limit of simple arcs \( L_n' \) enclosing one another. Therefore only multiple points of the "exterior adherent" type are possible. But we may verify that in the presence of one such point the region \( G_0'' \) may be enlarged while keeping to the rule of its construction, which contradicts the choice of \( G_0'' \).

Suppose that \( \bar{L} \) is the curve indicated above, \( A_1B_1 \) its endpoints on \( \overline{A_1A_0} \), \( \overline{B_0B_1} \), \( G_1'' \) the region enclosed by it along with \( A_1B_1 \). The distances from \( X \in \bar{L} \) to \( A_0B_0 \) satisfy (27). For \( \bar{L} \) each shortest arc from \( X \in \bar{L} \) to \( A_0B_0 \) going in \( P_1 \) passes within the closure \( \overline{G_1''} \).

For \( A_4B_4 \subset \bar{L} \) the closest points of the segment \( \overline{A_0B_0} \) are the endpoints \( A_0, B_0 \). Suppose that, counting from \( A_4 \) towards \( B_4 \), \( A_5 \) is the last point on \( \bar{L} \) for which the closest point on \( A_0B_0 \) is the point \( A_0 \), and, counting from \( B_4 \) towards \( A_4 \), \( B_5 \) the last point on \( \bar{L} \) for which the closest point on \( A_0B_0 \) is the point \( B_0 \). We join \( A_5 \) to \( A_0 \) with the rightmost of the shortest arcs \( \overline{A_5A_0} \) lying in \( G_1'' \) and \( B_5 \) with \( B_0 \) by the leftmost of the shortest arcs \( \overline{B_5B_0} \) lying in \( G_1'' \). These shortest arcs will not intersect among themselves, since \( \varepsilon < \rho' \). Further construction leads to a region \( G''' \), bounded by the contour \( \overline{A_0A_5B_5B_0} \).

4. Suppose that \( XM \) is an arbitrary shortest arc from the interior point \( X \) to \( A_0B_0 \). Then \( M \) lies inside \( \overline{A_0B_0} \). From subsection 4 of Chapter II, the angles which the shortest arc \( XM \) forms at the point \( M \) with the branches of the shortest arc \( \overline{A_0B_0} \) are not less than \( \pi/2 \). By the same principle, the sector angles of the region \( G''' \) at the vertices \( A_0, B_0 \) are
not less than \( \pi/2 \).

From each point \( X \in \overline{A_5B_5} \) we may draw one or several shortest arcs to \( \overline{A_5B_0} \) such that these shortest arcs lie in \( \overline{G'''} \). Suppose that \( XM \) and \( XN \) are two shortest arcs, with one of them lying to the right of the other in \( \overline{G'''} \). We are going to estimate the angle \( \alpha \) at the vertex of the triangle \( T \) which is formed by these shortest arcs and the segment \( MN \) of the shortest arc \( L \). Since the angles at the vertices \( M, N \) are not smaller than \( \pi/2 \), we have \( \alpha \leq \delta(T) \). Moreover, by Lemma 21 of Chapter V, we have \( \delta(T) \leq \omega^+(T) \), so that

\[
(28) \quad \alpha \leq \Omega(G) < \frac{1}{10}.
\]

In the construction on the plane of the triangle with the same lengths of its sides, for the angle \( \alpha_0 \) corresponding to \( \alpha \) we will have, from Lemma 23 of Chapter V,

\[
\alpha_0 \leq \alpha + 2\Omega(G) \leq \frac{3}{10},
\]

so that

\[
(30) \quad \rho(M, N) = 2\rho(X, M) \sin \frac{\alpha_0}{2} \leq \frac{3}{10} \rho(X, M) \leq \frac{3}{10} \varepsilon_3.
\]

Suppose that we have marked points \( Y_M, Y_N \) at distances of \( h = \varepsilon_3/3 \) from \( M \) and \( N \) on the shortest arcs \( XM, XN \). On swinging the triangle \( XY_MY_N \) onto the plane the angle \( \alpha_0' \) corresponding to \( \alpha \) may be estimated quite analogously to (29). Therefore, analogously to inequality (30), we will have

\[
(31) \quad \rho(Y_M, Y_N) < \frac{3}{10} \varepsilon_3 < h.
\]

5. We join each point \( X \in A_5B_5 \) to the corresponding point of \( M \) by the rightmost shortest arc to \( \overline{A_5B_0} \) lying in \( \overline{G'''} \). On the shortest arc \( XM \) we mark a point \( Y \) at a distance \( h \) from \( M \). If points \( X', X'' \) on \( \overline{A_5B_5} \) lying to the left and right of \( X \in \overline{A_5B_5} \) converge to \( X \), then the distance between the points \( Y', Y'' \) corresponding to them becomes less than \( 2h \) for sufficiently close \( X' \) and \( X'' \). This last situation makes it possible to choose on \( \overline{A_5B_5} \) a finite number of points \( X_i \) \((i = 1, 2, \ldots, m)\), to draw the rightmost shortest arcs \( X_iM_i \) to \( \overline{A_5B_0} \) and to mark on them points \( Y_i \) at distances \( h \) from \( \overline{A_5B_0} \), doing this in such a way that the condition

\[
(32) \quad \rho(Y_i, Y_{i+1}) < 2h \quad (i = 1, \ldots, m-1)
\]
is satisfied and the points $X_i$ and $X_m$ are arbitrarily close to $A_5$ and $B_5$ respectively.

The sequence of points $Y_i, Y_{i+1}$ ($i = 1, \cdots, m-1$) may be joined by shortest arcs. Because of (32) these shortest arcs cannot touch $A_0B_5$ and $A_5B_5$, so that they may be regarded as passing between the shortest arcs $XM_i$ and $X_{i+1}M_{i+1}$ drawn earlier, as in Figure 81.

If the points $M_i, M_{i+1}$ coincide, and $Y_i = Y_{i+1}$, then the shortest arcs just drawn bound a reduced convex triangle $M_i Y_i Y_{i+1}$. If $M_i \approx M_{i+1}$, then we obtain a convex quadrilateral $M_i M_{i+1} Y_{j+1} Y_j$, which may be decomposed into two reduced convex triangles by shortest arcs $M_j Y_{j+1}$ passing through it. If more than two successive points $M_i = M_{i+1} = \cdots = M_{i+t}$ coincide, then we will simply regard the shortest arcs $X_{i+1}M_{i+1}, \cdots, X_{i+t-1}M_{i+t-1}$ as dropped. In doing this condition (32) is not violated, since in this case $\rho(X_i, X_{i+1}) \leq 2h$. The case of equality is not possible, for otherwise there would be a complete angle at $M$ larger than $3\pi$, contradicting the condition (24).

6. We assert that the polygon $Y_1 Y_2 \cdots Y_m$ constructed in this way is, for sufficiently small initial quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, close to $L$ not only in position but in length. For the proof of this assertion we will compare the lengths $b_i$ of the pieces $Y_i Y_{i+1}$ with the lengths $a_i$ of the pieces $M_i M_{i+1}$.

In the case $a_i = 0$, i.e., when $M_i = M_{i+1}$, the angle $\alpha$ at the vertex $M_i$ of the triangle $T = M_i Y_{i+1} Y_i$ does not exceed $\omega (M_i)$, since the remaining portion of the complete angle encircling $M_i$ is itself not less than $2\pi$. On swinging $T$ onto the plane the corresponding angle $\alpha_0$, from estimate (74) of Chapter V, though possibly larger than $\alpha$, is not larger than it by more than $2\Omega (T)$. Therefore for $a_i = 0$

$$|b_i - a_i| = b_i = 2h \sin \frac{\alpha_0}{2} \leq h\alpha_0 \leq h[\omega (M_i) + 2\Omega (T)] \leq 3\Omega (T)h. \quad (33)$$

In the case $a_i \neq 0$, when $M_i \neq M_{i+1}$, the sector angle $\xi$ at the vertex $M_{i+1}$ in the triangle $T_1 = M_i M_{i+1} Y_{j+1}$ is equal to the angle $\xi$ between the sides of this sector and lies between $\pi/2$ and $\pi/2 + \omega (M_{i+1})$. On swinging
this triangle onto the plane this angle, from estimates (73) and (74) of Chapter V, changes by not more than $2\Omega(T_i)$. Therefore

$$\left| \xi_0 - \frac{\pi}{2} \right| \leq \omega^-(M_{j+1}) + 2\Omega(T_i) \leq 3\Omega(T_i).$$

Analogously for the angle $\tilde{\eta}$ of the sector at the vertex $M_j$ of the triangle $T_i$ and the angle $\tilde{\phi}$ of the sector at the vertex $M_j$ of the triangle $T_2 = M_j Y_{j+1} Y_j$ we obtain

$$|\tilde{\eta} - \eta_0| < 2\Omega(T_i), \quad |\tilde{\phi} - \phi_0| < 2\Omega(T_2).$$

Since, moreover,

$$\frac{\pi}{2} \leq \tilde{\phi} + \tilde{\phi} \leq \frac{\pi}{2} + \omega^-(M_t),$$

therefore

$$\left| \eta_0 + \phi_0 - \frac{\pi}{2} \right| < 2\Omega(T_i) + 2\Omega(T_2) + \Omega(M_t).$$

We apply the triangles $T_i, T_2$, developed on the plane, along the side $M_j Y_{j+1}$. Moreover, at the ends of the segment $M_j M_{j+1}$ lying in the plane, we erect perpendiculars $M_j M_j'$ and $M_{j+1} M_{j+1}'$ of length $h$, as shown on Figure 82.

From the quadrilateral $M_j Y_j Y_{j+1} M_{j+1}'$ we obtain

$$|b_j - a_j| \leq |M_j Y_j| + |Y_{j+1} M_{j+1}'|.$$ 

The terms on the right are easily estimated from the isosceles triangles $M_j Y_j M_j'$ and $M_{j+1} M_{j+1}' Y_{j+1}$:

$$|b_j - a_j| \leq 2h \sin \frac{\left| \eta_0 + \phi_0 - \frac{\pi}{2} \right|}{2} + 2h \sin \frac{\left| \phi_0 - \frac{\pi}{2} \right|}{2},$$

or, taking into account inequalities (34), (35),

$$|b_j - a_j| \leq 8h \Omega(T_i + T_2).$$

Adding all the inequalities (33) and (36) and noting that each point may attain a maximum in one estimate (33) and in two estimates (36), we obtain

$$\sum_{i=1}^m |b_i - a_i| \leq 19\Omega(G)h.$$
ROTATION OF SHORTEST ARCS

In view of the arbitrary smallness of \( h \), it follows that the length of the broken curve \( Y_1 \cdots Y_m \) is close to that of \( L \).

Lemma 2 is proved.

**Lemma 3.** If the endpoints \( A \) and \( B \) of the shortest arc \( L \) are joined by a broken curve \( L' \) which has no points other than endpoints in common with \( L \) and bounds together with \( L \) a region \( G \) homeomorphic to the disc, while the sector angles of the region \( G \) at the vertices \( A \) and \( B \) are different from zero, then the points \( A \) and \( B \) may be joined by a broken curve \( L \) which has no common points with \( L \) or \( L' \) other than \( A \) and \( B \), and differs arbitrarily little in length from \( L \).

**Proof.** We draw shortest arcs \( l_A, l_B \) of small length from the endpoints \( A \) and \( B \) into the sectors of the region \( G \). Moreover, we encircle the points \( A \) and \( B \) by simple closed polygons \( l_A', l_B' \) intersecting \( l_A \) and \( l_B \). From Lemma 2, we may construct simple polygons \( L'' \) differing little in length from \( L \), close to \( L \) and not intersecting it. We may also suppose that the \( L'' \) lie inside \( G \) and that their endpoints lie inside the contours \( l_A, l_B \).

From pieces of \( l_A, l_A', L'', l_B, l_B' \) it is easy to make up the broken curve \( L \) required in Lemma 3.

9. **Nonpositiveness of the rotation of a shortest arc.**

**Theorem 6.** The rotation of a shortest arc is nonpositive:

\[
\tau(L) \leq 0. \tag{38}
\]

**Proof.** We first suppose that the endpoints \( AB \) of the shortest arc lie at points where \( \theta \neq 0 \), so that the rotation \( \tau(L) \) exists.

We encircle \( L \) by a region \( G \) homeomorphic to the disc with absolute curvature satisfying

\[
\Omega(G - L) < \varepsilon. \tag{39}
\]

We join the endpoints of \( L \) by a simple polygon \( L_1 \) such that \( L_1 \) has no points other than endpoints in common with \( L \) and forms at \( A \) and \( B \) very small nonzero angles. Suppose that \( n \) is the number of vertices of \( L_1 \), \( \rho' \) the smallest distance between these vertices, \( \rho'' \) their smallest distance from the boundary of \( G \), and \( \rho''' \) the shortest distance of these vertices, excluding \( A \) and \( B \), from the shortest arc \( L \). We choose a quantity \( a > 0 \) very small in comparison with \( \rho', \rho'', \) and \( \rho''' \).

By Lemma 2 the endpoints of \( L \) may be joined by a broken curve \( L_2 \) passing between \( L \) and \( L_1 \) such that aside from the endpoints \( A \) and \( B \) the curve \( L_2 \) has no common points with \( L \) and \( L_1 \), and such that for the
lengths of the curve the inequality
\[ |s(L_2) - s(L)| < 4a \sin^2 \frac{\varepsilon}{4n} \]
holds, where \( \varepsilon \) is an arbitrary positive number.

In the polygon \( P \) which is bounded by the contour \( L_1 + L_2 \), we join the points \( A \) and \( B \) by the shortest of the curves \( \check{L} \) passing through \( P \). Evidently \( \check{L} \) is no shorter than \( L \) and no longer than \( L_2 \), so that
\[ (40) \quad |s(\check{L}) - s(L)| < 4a \sin^2 \frac{\varepsilon}{4n}. \]

The curve \( \check{L} \), as the shortest in the polygon, will be a broken curve which is stretched from within on certain vertices of the polygon \( P \). The sector angles at the vertices of \( \check{L} \), turned on the side away from \( L \), will be denoted by \( \phi_i \). At the vertices where \( \check{L} \) is stretched on \( L_2 \), the angles \( \phi_i \geq \pi \), for otherwise \( \check{L} \) could be shortened at \( P \). Therefore in the sum
\[ (41) \quad \sum_i (\pi - \phi_i) \]
the corresponding terms are nonpositive.

Suppose that \( C \) is a vertex at which \( \check{L} \) is stretched on \( L_1 \). We set aside on \( \check{L} \) on both sides of \( C \) segments \( CX_1, CX_2 \) of length \( a \). Since \( a \ll \rho', \rho'' \), the segments \( CX_1, CX_2 \) will lie on entire links of \( \check{L} \). Draw a shortest arc \( \overline{X_1X_2} \), forming along with \( CX_1, CX_2 \) a reduced triangle \( T = CX_1X_2 \). The triangles \( T \) thus constructed for each point \( C \), and even regions \( G_c \) containing them, which contain \( T \) along with the shortest arcs joining these points pairwise, will not overlap one another and will lie in \( (G-L) \), since \( a \ll \rho', \rho'' \). Therefore
\[ \sum_c \Omega(G_c) < \varepsilon. \]

The shortest arcs \( \overline{X_1X_2} \) may pass through \( C \) or on one or the other side of \( C \). If \( \overline{X_1X_2} \) passes through \( C \), then the term in the sum \( (41) \) corresponding to this point is nonpositive.

If \( \overline{X_1X_2} \) passes between \( C \) and \( L \), then on swinging the triangle \( T \) onto the plane, we will have
\[ \phi_0 \leq \pi, \quad \check{\phi} - \phi_0 \leq \omega^+(G_c), \quad 2\pi - \check{\phi} - \phi = \omega(C) \]
for the sector angle \( \check{\phi} \) at the vertex \( C \) and the angle \( \check{\phi} \) corresponding to the point \( C \). Adding these relations,
\[ (42) \quad \pi - \check{\phi} \leq 2\Omega(G_c). \]
We distinguish in general all the vertices $C$ at which condition (42) is satisfied. For these points the terms of (41) do not exceed $2\varepsilon$ in sum.

It remains to consider points $C$ at which $\pi - \phi \geq 2\Omega(G_c)$ and $\phi$ is the sector angle in the triangle $T$ itself. In view of estimate (72) of Chapter V, on swinging the triangle $T$ onto the plane we will have

\[ \pi - \phi_0 = \pi - \phi - (\phi_0 - \phi) \geq \pi - \phi - 2\Omega(G_c) > 0. \]

Denoting the length of $X_1X_2$ by $b$, from the isosceles triangle $T$ developed on the plane we find

\[ 2a - b = 2a \left( 1 - \sin \frac{\phi_0}{2} \right) = 4a \sin^2 \frac{\pi - \phi_0}{4} \geq 4a \sin^2 \frac{\pi - \phi - 2\Omega(G_c)}{4}. \]

We assert that for the triangles $T$ being considered here $\Sigma (\pi - \phi) \leq 3\varepsilon$. Indeed, otherwise we would have $\Sigma [\pi - \phi - 2\Omega(G_c)] > \varepsilon$, and for at least one of these vertices the inequality

\[ 2a - b > 4a \sin^2 \frac{\varepsilon}{4n} \]

would be satisfied.

Therefore the replacement of the portion $X_1CX_2$ of the polygon $L$ by the shortest arc $X_1X_2$ close to this vertex would shorten $L$ so much that it would become shorter than the shortest arc $L$, which is impossible.

Thus for the whole collection of points

\[ \Sigma (\pi - \phi) \leq 5\varepsilon. \]

Because of the arbitrary smallness of $\varepsilon > 0$, the arbitrary closeness of $L$ to $L$ and the arbitrary smallness of the angle between $L$ and $L$ at the endpoints, this proves the nonpositiveness of the rotation of $L$.

Proceeding with the proof, we have been supposing that the endpoints $A$ and $B$ of the shortest arc $L$ lie at points with nonzero complete angles (there are certainly no such points on the interior of the shortest arcs). Now this requirement may be removed. Since the rotation of any segment $A'B'$ of the shortest arc $L$, lying inside $AB$, is nonpositive, and since on the approach of $A'$ and $B'$ to $A$ and $B$ respectively it varies monotonically, and since from Theorem 3 and its nonpositiveness it is bounded, therefore the limit

\[ \lim_{A' \to A, B' \to B} \tau(A'B') \]

exists.

If it was the case that the complete angles at the points $A$ and $B$ were
nonzero or more generally if the construction of the polygons \( l_n \) necessary in definition (7) was realizable, then the limit (44) coincides with the rotation \( \tau(AB) \).

Otherwise, the limit (44) may be considered as a definition of \( \tau(AB) \) and this limit will turn out to be nonpositive for any shortest arc.

4. Pasting together of polygons.

10. Excision of a polygon. Suppose that \( R \) is a two-dimensional manifold with metric \( \rho \). Distinguish in \( R \) an open connected region \( G \), each two points \( A, B \) of which may be joined by at least one arc \( \widehat{AB} \) of finite length lying in \( G \). As we have already noted in subsection 3 of Chapter I, the equation

\[
\rho_G(A, B) = \inf_{\widehat{AB} \subset G} [s_{\rho} \widehat{AB}]
\]

defines in \( G \) an induced metric \( \rho_G \). If the region \( G \) were convex, then \( \rho_G = \rho \). In the contrary case \( \rho_G \geq \rho \). However, in a sufficiently small neighborhood of each point \( \rho_G = \rho \). Therefore the length of curves lying in \( G \) is the same in the metrics \( \rho_G \) and \( \rho \). This makes it possible in the right side of (45) to measure the length \( \widehat{AB} \) in the metric \( \rho_G \). Then (45) shows that the induced metric \( \rho_G \) is intrinsic.

If instead of an open region \( G \) we select a closed region \( \overline{G} \), each two points of which may be joined in \( \overline{G} \) by a curve of finite length, then the metric \( \rho_G \) is defined analogously to (45). The presence of a boundary on \( \overline{G} \) may lead to peculiarities of an essential nature in the structure of the distinguished metric space \( R_G \). Thus, even in the case when \( \overline{G} \) is bounded by a simple closed curve, it can happen that the metric space \( R_G \) may differ in its topological structure from the topological space which \( G \) represents as a subset of \( T \).

Example. Suppose that \( R \) is a plane, \( \Omega \) a point on the boundary of \( \overline{G} \) and near \( \Omega \) on the boundary of \( \overline{G} \) there are protuberances converging to \( \Omega \) in the form of "sleeves" of constant length 1 and twisted in ever smaller spirals. Then the endpoints \( \Omega_n \) of these sleeves converge to \( \Omega \) in \( R \), and in \( R_G \) they remain at distances not less than unity from the point \( \Omega \).

In this section we are interested in the simple case when a polygon \( \overline{P} \) is distinguished from a two-dimensional manifold \( R \) of bounded curvature. We are supposing that \( P \) is closed relative to this manifold, i.e., \( P \) is a compact connected region bounded in \( R \) by a finite number of simple polygonal lines. If we distinguish the connected polygon \( \overline{P} \) in \( R \), the
possibility of joining any two points of \( \bar{P} \) by a curve of finite length running in \( \bar{P} \) is automatically fulfilled. Therefore the metric \( \rho_P \) exists.

The distinguishing of the polygon \( \bar{P} \) in \( R \) and the consideration of \( \bar{P} \) as a metric space \( R_P \) with metric \( \rho_P \) is called *excision* of the polygon \( \bar{P} \) from \( R \).

The following assertions concerning the excision of a polygon from a two-dimensional manifold of bounded curvature are valid.

**Lemma 4.**

1. Close to an interior point of the region \( \bar{P} \), and also close to each point of the boundary of \( \bar{P} \) which is not a vertex of the polygon \( \bar{P} \), the metrics \( \rho \) and \( \rho_P \) coincide.

If at the vertex \( A \) of the polygon \( \bar{P} \) the sector of \( \bar{P} \) adjacent to \( A \) is convex, then also close to this vertex \( \rho_P = \rho \). If the sector is nonconvex, then \( \rho \) and \( \rho_P \) may differ in an arbitrarily small neighborhood of the vertex \( A \).

2. The space \( R_P \) with the metric \( \rho_P \) has the same topology as \( \bar{P} \) had as a subset of \( R \), i.e., it has the same collection of open sets.

3. Curves running in \( \bar{P} \) are simultaneously rectifiable or nonrectifiable in the metrics \( \rho_P \) and \( \rho \). In the first case their lengths in these metrics coincide. Therefore it also follows that the metric \( \rho_P \) is intrinsic. We shall not stop to prove Lemma 4.

In a considerable number of cases we shall be interested in polygons homeomorphic to the closed disc. However we do not exclude from consideration compact polygons of other topological structure, bounded by several simple closed polygons or having no boundary at all, which is possible when \( R \) is a closed manifold and \( \bar{P} \) coincides with all of \( R \). We may consider also polygons bounded by *geodesic* polygons. In the case of a compact polygon this gives nothing new, since a closed geodesic may, by introducing on it a finite number of new vertices, be converted into a broken curve consisting of shortest arcs.

We may also consider bounded but noncompact polygons which may retain an incompleteness inherited from the space \( R \). For example, if \( R \) is an open plane strip and we excise from it a “polygon” \( \bar{P} \) as in Figure 83 by means of geodesic polygons, whose “endpoints” rest on the boundary of the strip and therefore do not belong to \( R \) nor to \( \bar{P} \), then the
distinguished polygon may have the structure of a manifold with an incomplete boundary.

Finally one may take into consideration infinite polygons which are bounded in $R$ by simple polygonal curves, adding the single requirement that the vertices of the boundary curves do not have condensation points in $R$.

For all such connected manifolds the operation of excision retains its meaning and Lemma 4 is not affected.

Analogously, in $R$ one may distinguish polygons with cuts along certain simple polygons, as described in subsection 15 of Chapter I.

The operation of excision of a polygon can serve as a way of defining certain two-dimensional metrized manifolds with an edge.

11. \textit{Augmentation of a space.} Suppose that the metric space $R$ is not complete, i.e., there exists in it a fundamental sequence of points which have no limit point in $R$. Then $R$ may be augmented by adding to $R$ the limit points of such sequences. Sequences whose points become arbitrarily close to one another have a common limit point. The augmented space is a metric space $R'$ if the distance to the newly introduced points is defined as the limit of distances to the points of the corresponding sequence.

Augmentation of a two-dimensional manifold of bounded curvature may be one of the ways of constructing two-dimensional metrized manifolds with edges.

We note a simple case when the augmentation operation gives spaces already mentioned earlier.

\textbf{Lemma 5.} 1. Suppose that $R$ is a complete two-dimensional manifold of bounded curvature, $\bar{P}$ the interior portion of a connected polygon $P$ excised from $R$, $R_\bar{P}$ the space with metric $\rho_{\bar{P}}$ induced by excision of the closed polygon $\bar{P}$, $R_P$ the space with metric $\rho_P$ induced by excision from $R$ of the open polygon $P$, and $R'_P$ the augmentation of the space $R_P$. Then the spaces $R_\bar{P}$ and $R'_P$ are isometric.

2. Suppose under the same conditions that $\bar{P}$ is a closed polygon with cuts, and $P$ its interior portion (after removal of the cut lines). Then the spaces $R_P$ and $R'_P$ are isometric.

The proof of this lemma follows from the assertion of Lemma 2 that any shortest arc $L$, a piece of the side of $\bar{P}$ or $\bar{P}$, may be approximated by a broken curve passing arbitrarily close to $L$ and with a length arbitrarily close to $L$, and at the same time lying entirely on one side of $L$. 
thus inside $\overline{P}$ or $\overline{P}$.

According to Lemma 5 for open polygons excised from complete two-dimensional manifolds of bounded curvature, the operation of subsequent augmentation gives nothing new in comparison with direct excision of closed polygons.

Remark. Generally augmentation of two-dimensional manifolds of bounded curvature may lead to more complicated spaces. We give two examples.

1) Suppose an infinite sequence of conical protuberances is situated on the plane, converging to the point $O$. In terms of its intrinsic geometry such a surface, on removal of the point $O$, becomes a two-dimensional manifold $R$ of bounded curvature. Augmentation of $R$ adjoins the point $O$ to $R$, and in an arbitrarily small neighborhood of this point the positive and negative parts of the curvature of $R$ are infinite.

2) Consider the infinite-sheeted plane $R$ with branch point $O$. The point $O$ is not counted as a point in $R$. $R$ is a manifold of zero curvature. Its augmentation attaches the point $O$ to $R$. There then appears around $O$ an "infinitely large sector", which cannot appear in a space with a metric induced by the distinguishing of a set on a two-dimensional manifold of bounded curvature.

In the paper [31] the class of augmented spaces $F'$ for which the following three conditions are satisfied, was investigated: 1) the original space $F$ is a two-dimensional manifold of bounded curvature, where the curvature of bounded sets in $F$, including noncompact bounded sets, is finite; 2) $F'$ is topologically a two-dimensional manifold with an edge; 3) the augmentation $F' - F$ consists only of the edge $F'$. In [31] Ju. F. Borisov investigated the intrinsic geometry of such $F'$-spaces and proved for them a series of theorems on excisions and pastings.

12. Pasting of polygons. In subsection 16 of Chapter I we described an operation of pasting metrized manifolds $\overline{R}_j$ with edges. Here the metric $\rho$ of the pasted manifold $R$ was defined as

$$\rho(A, B) = \inf_{Z(A, B)} \left[ \sum_{i} \rho_{R_j}(X_i, X_{i+1}) \right],$$

where $Z(A, B)$ are finite chains of points $A = X_1, X_2, \ldots, X_n = B$, in which each pair of successive points $X_i, X_{i+1}$ belongs to one of the pasted manifolds $\overline{R}_j$.

If all the pasted manifolds $\overline{R}_j$ were connected polygons with boundary,
excised from two-dimensional manifolds of bounded curvature, and if moreover on pasting each time the identified portions of the boundary have correspondingly equal lengths, then in this case the metric of the pasted space may be defined by the equation

$$\rho(A, B) = \inf_{L(A, B)} [s(L)],$$

where $L(A, B)$ is an arbitrary curve joining the points $A$ and $B$ in the pasted space $R$. The length $s(L)$ of the curve $L(A, B)$ is defined as follows. The curve $L(A, B)$ is decomposed into a countable set of open pieces, running inside the separate sets $\overline{R}_i$, and the remaining portion of $L(A, B)$. The length of the distinguished pieces is measured in the metric of the corresponding original manifolds $\overline{R}_i$, the length of the whole curve being supposed infinite if any one of these pieces is not rectifiable. The remaining portion of $L(A, B)$ is a finite number of closed sets lying on separate ribs of the pasting. The length of each of these sets is defined as the linear measure of this set on the corresponding rib of the pasting. Because portions of equal length were identified in the pasting, this last definition has a definite meaning.

**Lemma 6.** If on pasting the polygons $\overline{R}_i$ the pasting is carried out on pieces of the boundary of correspondingly equal length, then the definition (47), which here has a meaning, leads to the same intrinsic metric $\rho$ to which definition (46) leads in this case.

**Proof.** The inequality

$$\inf \left[ \sum_{Z(A, B)} \rho_j(X_i, X_{i+1}) \right] \geq \inf_{L(A, B)} [s(L)]$$

is valid because each pair of points $X_i, X_{i+1}$ may be joined by a curve lying, with the exception of its endpoints, strictly inside $\overline{R}_i$, and of almost the same length as $\rho_j(X_i, X_{i+1})$.

The inequality

$$\inf \left[ \sum_{Z(A, B)} \rho_j(X_i, X_{i+1}) \right] \leq \inf_{L(A, B)} [s(L)]$$

is valid in view of the following considerations. If $s(L)$ is finite, then $L$ has common points with a finite number of ribs of the pasting. If $L$ intersects various ribs at points arbitrarily close to the vertices of the pasting $M$, then it may be removed by changing the curve $L$ arbitrarily little. It is sufficient to replace the portion of $L$ close to $M$ by two segments of ribs running in $M$. After this the lengths of the pieces of $L$
running from one rib to another, and the endpieces of \( L \) from \( A \) and \( B \) to the closest points to them on \( L \) which are intersections of \( L \) with the ribs may, with arbitrarily high precision, be replaced by the "lengths" of the chains with the same endpoints. Each of the remaining pieces of \( L \) goes from one point of a rib to another point of the same rib and does not leave the limits of the two polygons pasted along that rib. The length of such a piece does not increase if it is replaced by a portion of the rib, or, what is the same thing, by a chain consisting in all of two points—the endpoints of that piece. Thus, for any \( \varepsilon > 0 \) we may for \( L(A,B) \) construct a chain \( Z(A,B) \) for which

\[
\sum_i \rho_i(X_i, X_{i+1}) < s(L) + \varepsilon.
\]

This proves inequality (49). From (48) and (49) it follows that Lemma 6 is valid.

In considering the operation of pasting polygons, we have imposed a supplementary restriction consisting in the requirement that on any segment of the boundary the pasting is realized along pieces of equal length. This restriction is not by chance. In the first place we are interested in the case when the pasting is carried out on a manifold of bounded curvature. But the following assertion holds, proving the well-known necessity of the hypothesis which we have made.

**Lemma 7.** If a pasting of manifolds, which have been excised from manifolds of bounded curvature, leads again to a manifold of bounded curvature, then the pasting can be accomplished on pieces of equal length.

We shall explain the proof of this lemma. It makes use of one of the results of Chapter IX.

The length of every simple curve is not greater than the lower limit of the lengths of polygons converging to the curve from one side of it. Under the conditions of Lemma 7 every small segment \( L \) of the rib being pasted was a shortest arc in the original manifolds, and, by Lemma 2, had a length equal to the lower limit of the lengths of polygons converging to \( L \) from one side. Moreover, from Theorem 6, the rotation of \( L \) on each section was nonpositive. If the pasting was applied to a manifold of bounded curvature, then the rotation of the sections of \( L \) was preserved. Therefore \( L \) turns out again in the new space to be a so-called curve with rotation of bounded variation, and does not contain inside itself any points with \( \theta = 0 \). Then, from Theorems 3 and 5 of Chapter IX, the
length of each portion of \( L \) in the new space is finite and also is equal to the lower limit of the lengths of polygons converting to that portion from one side. But the length of polygonal curves lying inside the polygons surely remains invariant under the pasting. Therefore the new length of a section \( l \) of the curve coincides with its length in both of the original manifolds, i.e., the pasting has been accomplished along sections of equal length.

13. **First theorem on pasting.**

**Theorem 7.** If from the polygons \( \tilde{P}_j \), each of which was excised from some two-dimensional manifold \( R_j \) of bounded curvature, there has been pasted a two-dimensional manifold \( R \), the pasting of the edges having been done along sections of correspondingly equal length, then the resulting manifold \( R \) itself is a two-dimensional manifold of bounded curvature.

(Since \( R \) is a manifold, all the ribs of polygons counted with \( \tilde{P}_j \) are subjected to pasting.) Among the polygons \( \tilde{P}_j \) we may understand geodesic polygons and polygons with cuts.

We precede the proof of Theorem 7 by two Lemmas.

**Lemma 8.** Under the hypotheses of Theorem 7 each rib \( l \) of the pasting is a geodesic in the manifold \( R \), i.e., is a shortest arc in a small neighborhood of each fixed point on it. The length of \( l \) in \( R \) coincides with the length of \( l \) in the metrics of the original manifolds whose pieces were pasted along \( l \).

**Proof.** 1. Suppose that the polygons \( \tilde{P}_1, \tilde{P}_2 \) were pasted along \( l \). Suppose that these were excised from manifolds with the metrics \( \rho_1, \rho_2 \). Let \( O \) be any point inside \( l \). Consider a piece \( MN \) of the rib \( l \) containing \( O \), which is a shortest arc in the metrics \( \rho_1, \rho_2 \). Single out a segment \( AB \) of the rib \( l \) containing \( O \). We take the segment \( AB \) so small that its diameter in the metric \( \rho \) of the pasted manifold \( R \) is less than the distance of the segment \( AB \) in the metric \( \rho \) to the boundaries of \( P_1 \) and \( P_2 \), if the piece \( MN \) is excluded from the latter. Then, as is easily established from the definition (47), the distance \( \rho(A,B) \) coincides with the length of \( AB \) in the metrics \( \rho_1 \) and \( \rho_2 \).

2. Since the last assertion is valid for each piece of the segment \( AB \), it therefore follows that \( AB \) is a shortest arc in the metric \( \rho \) and the length of \( AB \) in the metric \( \rho \) coincides with the length of \( AB \) in the metrics \( \rho_1, \rho_2 \).

3. From the local result just obtained it follows that the full length of the rib \( l \) is finite and coincides with the length of \( l \) in the metrics \( \rho_1, \rho_2 \).
4. It remains to be proved that on a sufficiently short portion near an endpoint, the rib \( l \) also is a shortest arc in the metric \( \rho \).

Suppose that the point \( O \) is a vertex of the pasting and an endpoint of the rib \( l \). On all the ribs \( l_i \) \((i = 1, 2, \cdots, n)\) which issue from \( O \) (we are taking \( l = l_i \)) we mark off segments \( OM_i \) inside whose limits the \( l_i \) remain shortest arcs in the corresponding original metrics. On the piece \( OM_i \) of the rib \( l_i \) we choose a segment \( X_iO \) whose diameter in the metric \( \rho \) is less than the distance from \( X_iO \) to the boundaries of the polygons adjacent to \( O \), excluding from the latter all the pieces \( OM_i \).

Because of the local compactness of the manifold \( R \) the point \( X_i \) may be joined in \( R \) by a shortest arc \( a \) to the point \( O \). The shortest arc \( a \) may intersect several ribs \( l_i \). If on the section between two points of intersection of \( a \) and \( l_i \), the shortest arc \( a \) does not encounter other ribs \( l_i \), then the corresponding piece of \( a \) will be replaced by a piece of \( l \). This does not lengthen \( a \) and therefore leaves \( a \) a shortest arc. Outside any neighborhood of the point \( P \) the shortest arc \( a \) has a finite number of passages from one rib to another. If \( a \) goes from one rib \( l_i \) to the next, and then without intersecting other ribs returns to \( l_i \), we may also replace the corresponding piece of \( a \) by a segment of \( l_i \). After these transformations either \( a \) will coincide with \( \overline{X_iO} \), in which case their lengths are equal and \( \overline{X_iO} \) is a shortest arc, or else the shortest arc \( a \) will pass from one rib to the following one, encircling the point \( O \) along a spiral with a finite or infinite number of turns. Mark a single point in the intersection of \( a \) with each successive rib \( l_i \). Evidently in each sector we will have

\[
\overline{X_iO} \leq X_iX_{i+1} + \overline{X_{i+1}O}.
\]

Therefore

\[
\overline{X_iO} \leq X_iX_2 + X_2X_3 + \cdots = a.
\]

But inasmuch as \( a \) is a shortest arc, then \( \overline{X_iO} \geq a \), which along with the preceding yields \( \overline{X_iO} = a \). Thus in this case \( \overline{X_iO} \) is a shortest arc.

Lemma 8 is completely proved.

**Lemma 9.** If under the hypotheses of Theorem 7, in the pasted space \( R \) there issue from the point \( O \) a rib of the pasting \( l \) and a shortest arc \( a \), intersecting at points arbitrarily close to \( O \), then the upper angle (in \( R \)) between \( l \) and \( a \) at the point \( O \) is equal to zero.

**Proof.** 1. We will suppose that \( O \) is a vertex of the pasting. (If \( O \) lies inside a rib, we include the point \( O \) among the vertices. Then two
ribs converge at it.) Suppose that \( l_1 = l, l_2, \ldots, l_n \) are the ribs of the pasting converging in \( O \). We carry out our discussion in a neighborhood of the point \( O \) where all the ribs \( l_i \) are themselves shortest arcs.

2. The common portion of \( l \) and \( a \) is a closed set \( f \) on \( l \). On the intervals of the complement of \( f \) the shortest arc \( a \) departs from \( l \) and then returns to \( l \). The lengths of the corresponding segments of \( a \) are equal to the lengths of the omitted piece \( s \) of \( l \), since \( l \) and \( a \) are shortest arcs to the point \( O \).

We may divide the complementary intervals mentioned above into four types: intervals on which \( a \) leaves and returns to \( l \) on the right without circling the vertex \( O \), analogous intervals with \( a \) leaving to the left, intervals on which \( a \) leaves \( l \) from the right and returns to \( l \) on the other side, going around the point \( O \), and analogous intervals with \( O \) encircled in the opposite direction.

Consider in place of \( a \) four other shortest arcs: a shortest arc \( a_1 \) which coincides with \( a \) on pieces of the first type, and on the remaining pieces goes along \( l \); a shortest arc \( G_2 \) which coincides with \( a \) on pieces of the second type, and on the remainder goes along \( l \); a shortest arc \( a_3 \) which coincides with \( a \) on pieces of the third type and on the remainder goes along \( l \); analogously a shortest arc \( a_4 \). If we verify that \( a_1, a_2, a_3, a_4 \) form in the metric \( \rho \) of the space \( R \) a null upper angle at the point \( O \), then this will be valid also for the shortest arc \( a \).

Evidently it is sufficient to consider the angles between \( a_1, l \) and \( a_3, l \).

3. We shall move along \( a_1 \) towards \( O \). If beginning at some place \( a_1 \) coincides with \( l \), then the upper angle \( \alpha(a_1, l) = 0 \). Suppose that arbitrarily close to \( O \) there is a piece where \( a_1 \) departs from \( l_1 = l \) in the sector \( Q_1 \) between \( l_1 \) and \( l_2 \), but does not intersect \( l_2 \). Then

\[
\alpha(l, a_1) \leq \alpha_{Q_1}(l, a_1) = 0.
\]

This last equation is valid because it follows from the equation \( \alpha_{R_1}(l, a_1) = 0 \) in the manifold \( R_1 \) from which the polygon with the sector \( Q_1 \) was excised, and from the last point of Theorem 6 of Chapter IV, that the angle \( \alpha \) measured in \( Q_1 \) between \( l \) and \( a_1 \) is also equal to zero.

Suppose finally that arbitrarily close to \( O \) the shortest arc \( a_1 \) departs from \( l \), intersecting some number of ribs \( l_2, \ldots, l_m \), and returns to \( l_1 \). In this case

\[
\alpha(l_1, l_2) = \alpha(l_2, l_3) = \cdots = \alpha(l_{m-1}, l_m) = 0.
\]

since for \( i = 1, \ldots, m - 1 \)
\[
\tilde{\alpha}(l_i,l_{i+1}) \leq \tilde{\alpha}_{Q_i}(l_i,l_{i+1}) = 0.
\]

Consider the shortest arcs \(a_i\) \((i = 1, \cdots, m)\), where \(a_i\) coincides with \(a_i\) in the sector \(Q_i\) between \(l_i\) and \(l_{i+1}\) and with \(l_i\) or \(l_{i+1}\) on the remaining portions. Then we have

\[
\tilde{\alpha}(l_i,a_i) \leq \max[\tilde{\alpha}(l_i,a_i); \tilde{\alpha}(l_i,a_i^2); \cdots; \tilde{\alpha}(l_i,a_i^n)] \\
\leq \max[\tilde{\alpha}(l_i,a_i); \tilde{\alpha}(l_i,l_2) + \tilde{\alpha}(l_2,a_i^2); \tilde{\alpha}(l_i,l_2) + \tilde{\alpha}(l_2,l_3) + \tilde{\alpha}(l_3,a_i^3); \cdots] = 0.
\]

Thus, in all cases \(\tilde{\alpha}(l,a_i) = 0\).

4. We move along \(a_3\) towards 0. If, beginning with some position, \(a_3\) coincides with \(l\), then \(\tilde{\alpha}(l,a_3) = 0\). Suppose that this were not so and that arbitrarily close to 0 there is a piece where \(a_3\) goes around the point 0 (Figure 84). In this case

\[
\tilde{\alpha}(l_1,l_3) = \tilde{\alpha}(l_2,l_3) = \cdots = \tilde{\alpha}(l_n,l_1) = 0
\]

and each of the sectors \(Q_i\) \((i = 1, \cdots, n)\) has in the original manifold \(R_i\) from which it was excised before the pasting a zero sector angle.

If the complete angle at the vertex \(Q_i\) in \(R_i\) is nonzero, then the sector \(Q_i\) is convex in \(R_i\). In a sufficiently small neighborhood of the vertex of such a sector each of its points may be joined in \(R_i\) with its sides passing in \(Q_i\) by shortest arcs which are very short in comparison with the distance of the point to the vertex of the sector. This is possible according to Theorem 11 of Chapter IV. If the complete angle at the vertex \(Q_i\) in \(R_i\) was zero, then through each point on a shortest arc, which passes in the sector \(Q_i\) from its vertex, there may be drawn a loop enclosing the vertex of the sector and very short in comparison with the distance from that point to the vertex (see Lemma 17 of Chapter IV).

Now suppose that \(X\) and \(Y\) are arbitrary points on \(l\) and \(a_3\), very close to 0, with \(OY \leq OX\). On the basis of what has just been said, it is possible to pass through the point \(Y\) in the whole manifold \(R\) a loop enclosing \(O\), very short in comparison with \(OY\). Suppose that \(X'\) is the point where this loop intersects that one of the shortest arcs \(l, a_3\) on which the point \(X\) lay. Then \(\gamma(X,X') = 0\). But \(\gamma(X,Y)\), as in the last case of Theorem 12 of Chapter IV, is very little different from \(\gamma(X,X')\). Therefore
\[ \limsup_{x, y \to 0} \gamma(X, Y) = \bar{\alpha}(l, a_3) = 0. \]

Shortest arcs of types \( a_2, a_4 \) do not differ from \( a_1, a_3 \), so that the proof of Lemma 9 is complete.

We turn to Theorem 7 itself. In order to verify that \( R \) is a manifold of bounded curvature, it is sufficient to show that at each point \( O \subset R \) there is a neighborhood \( G \) within whose limits the sum of the excesses, taken with respect to the upper angles, is uniformly bounded above for all finite systems \( \{T_i\} \subset G \) of nonoverlapping convex triangles homeomorphic to the disc:

\[
\sum_i \bar{\delta}(T_i) \leq N(G).
\]

If \( O \) is an interior point of one of the \( \bar{P}_j \) being pasted, then this assertion is trivial, since \( \bar{P}_j \) is excised from a manifold of bounded curvature.

Suppose that \( O \) lies on a rib of the pasting. Without loss of generality, we may consider \( O \) to be a vertex of the pasting. Take a neighborhood \( G \) of the point \( O \) not containing other vertices, in which pieces of the ribs of the pasting are shortest arcs in \( R \) and in which there fall compact pieces \( P_k \) of the pieces of \( \bar{P}_j \) adjacent to \( O \). We assert that in this case estimate (50) holds with

\[
N(G) = 2\pi + \sum_k \omega_k^+(P_k).
\]

We shall carry out the proof in the following order. Beginning with any of the systems \( \{T_i\} \) mentioned above, we transform each of the triangles \( T_i \) into \( T'_i \) in such way that their excesses do not decrease, \( T'_i \) remain nonoverlapping, convex, homeomorphic to the disc, and the ribs of the pasting decompose \( T'_i \) into only a finite number of polygons \( Q \). The polygons \( Q \) are decomposed into triangles \( t \), not decreasing the sum of the excess or decreasing it by no more than \( 2\pi \). The total excesses of all the triangles \( t \) will not in this process exceed \( \sum_k \omega_k^+(P_k) \).

Thus, suppose that \( \{T_i\} \) is a finite system of nonoverlapping convex triangles homeomorphic to the disc. We consider a particular triangle \( T_i \).

1. If \( T_i \) belongs to one of the pieces \( P_k \), we drop \( T_i \) without change.
2. If \( T_i \) does not lie in any piece \( P_k \), then no rib of the pasting intersects the side of \( T_i \) arbitrarily close to a vertex of \( T_i \), so that we may pick out terminal portions of the sides of \( T_i \) each one of which lies in one of the pieces \( P_k \). The remaining pieces of the sides of \( T_i \) lie at finite distances in \( R \). Therefore each rib of the pasting intersecting \( T_i \) has
a finite number of passages from one side to another. On the remaining
pieces the rib of the pasting may intersect several times one side of $T_i$. On each such segment we replace pieces of the side of $T_i$ by pieces of the rib of the pasting each time when the latter arrives inside $T_i$. Here the angles $T_i$ do not change, and $T_i$ will be decomposed by the ribs of the pasting into a finite number of pieces.

3. If at least one of the ribs $l$ of the pasting intersects both sides of a triangle $T_i$ with vertices $A,B,C$ arbitrarily close to the vertex $A$, then, from Lemma 9, these sides form with $l$, and accordingly with one another, a zero angle at the point $A$. In this case we may, without decreasing the excess of $T_i$, choose one of the pieces $MN$ of the rib $l$ going from $AB$ towards $AC$ and replace $BAC$ by $BNC$ (Figure 85).

4. If the preceding case does not hold, but some rib $l$ of the pasting intersects one side $AB$ arbitrarily close to $A$, then $l$ forms at $A$ a zero angle with $AB$. Without changing the excess of the triangle we may then replace the pieces of the side $AB$ close to $A$ by the pieces of $l$ entering into $T_i$.

Using the construction just described, we convert $\{T_i\}$ into a system $\{T'_i\}$ with no decrease in the sum of the excesses, with the $T'_i$ being decomposed by the ribs of the pasting into a finite number of polygons $Q$. Each polygon $Q$ lies in one of the $P_k$.

The excesses of the triangles $T'_i$ do not decrease if they are counted not with respect to the upper angles in the metric $\rho$ between the sides $T'_i$ but rather with respect to the upper angles measured inside their sectors. The latter is not greater than the sum measured in $\bar{R}_k$ of the angles of those sectors from which the given sector was made up at the vertex of $T'_i$. In what follows we shall examine the excesses with respect to this last index.

Suppose that the point $O$ does not lie inside the triangle $T_i$. In the process of converting $T_i$ into $T'_i$ there occurred a division by shortest arcs. Therefore all the $Q$ into which the $T'_i$ is subdivided are convex. We shall decompose them by diagonals into reduced triangles $t$. The sum of the excesses of all the $t$ will be not less than the excess of $T'_i$ and therefore
not less than \( \delta(T_i) \).

An exception might be constituted by the single triangle \( T_0 \) within which the point \( O \) lies. It will be decomposed into convex polygons \( Q \), one of which in its turn is decomposed into polygons by the ribs of the pasting which enter into \( O \). These last polygons may fail to be convex. We decompose them by diagonals drawn from the point \( O \). For \( T_0 \) the sum of the excesses of the resulting triangles \( t \) may turn out to be less than the excess of \( T_0 \) by \( 2\pi - \theta \), where \( \theta \) is the total angle of the sectors pasted around \( O \), i.e., it is not larger than \( 2\pi \).

Thus finally
\[
\sum_t \delta(T_i) \leq 2\pi + \sum \delta_r(t) \leq 2\pi + \sum \omega_{\kappa_i}(P_i),
\]
which proves Theorem 7.

From Theorem 7 and the very character of the concept of a sector angle there follows the following assertion.

**Theorem 8.** On pasting polygons excised from manifolds of bounded curvature, the rotations of the pieces of the ribs of the pasting from the corresponding sides and the sector angles at the vertices of the pasting remain as before.

Therefore, in particular, after the pasting the curvature on a rib is equal to the sum of the rotations which this rib had in the pasted polygons, and the curvature at a vertex is \( 2\pi - \theta \), where \( \theta \) is the sum of the sectors of the original polygons pasted around this vertex.

5. **Estimation of the excesses and the distortion of angles in terms of the curvature.**

14. **Excess and curvature of a polygon.** Consider a closed geodesic polygon \( P \). Suppose that \( \chi(P) \) is the Euler characteristic of this polygon, and \( L_1, \ldots, L_n \) closed geodesic broken curves bounding \( P \) (\( n \) may be zero).

From subsection 7 on the Gauss-Bonnet theorem, for the rotations of the boundaries \( L_k \) from the side of \( P \) and for the curvature of the interior portion \( P_- \) of the polygon \( P \) the relation
\[
\omega(P_-) = 2\pi \chi(P) - \sum_{k=1}^n \tau(L_k)
\]
holds.

Therefore it follows that for the excess of the polygon \( P \), defined by the relation
Theorem 9.

$$
\delta(P) = \omega(P) + \sum \tau_i,
$$

where the \( \tau_i \) are the rotations of the ribs of the bounding polygonal curves from the side of the polygon \( P \), and \( P_\) is the interior portion of the polygon \( P \).

Separating out the positive and negative parts of the curvature in (52) and noting that all the \( \tau_i \leq 0 \) from Theorem 6, we may rewrite (52) in the form

$$
\delta(P) = \omega^+(P) - \omega^-(P) - \sum \tau_i^-.
$$

This equation implies the estimate

$$
-\omega^-(P) - \sum \tau_i^- \leq \delta(P) \leq \omega^+(P).
$$

Finally we suppose that the polygon \( P \) has been excised from the manifold in question, and that we have pasted to it along each boundary \( L \) the lateral surface of a right circular cylinder with the same perimeter on the base as the length of \( L \). For the resulting manifold \( R \) the curvature \( \omega(P) \) remains as before, and the curvature of the boundary ribs of \( P \) will coincide with the rotation of these ribs from the side of \( P \). Therefore equation (52) takes the form

$$
\delta(P) = \omega_R(P - \sum A_i),
$$

where the \( \alpha_i \) are the sector angles at the vertices of the polygon \( P \), the following theorem holds.

$$
\delta(P) = 2\pi \chi(P) - \sum_j (\pi - \alpha_j),
$$
where the $A_i$ are the vertices of $P$.

15. Distortion of the angle of a triangle on swinging it onto the plane. A reduced geodesic triangle lying in a region homeomorphic to the disc is called a normal triangle if no two of its vertices may be joined in it by a curve which is shorter than the corresponding side of the triangle.

Theorem 10. Suppose that $T$ is a normal triangle, $\alpha$ an angle, $\bar{\alpha}$ the same angle measured in the sector of the triangle, $\alpha$ the corresponding sector angle, and $\alpha_0$ the angle in a plane triangle with sides of the same length. Then the estimate

$$-\omega^-(T_-) - \sum_{i=1}^{3} \tau_i^- \leq \bar{\alpha} - \alpha_0 \leq \bar{\alpha} - \alpha_0 \leq \omega^+(T_-)$$

is valid, where $T_-$ is the interior portion of the triangle $T$, $\tau_i^-$ the negative parts of the rotations of the sides of $T$ from the side reverse to the interior region of $T$. If the triangle $T$ has interior tails, then in the computation of $\sum \tau_i^-$ the rotation of this tail is taken into account from both sides. If $T$ has an exterior tail, then the rotation of the latter is taken equal to the sector angle in the triangle at the base of this tail and it is counted once in the calculation of $\sum \tau_i^-$. We recall that in the presence of an interior tail its points are not considered as interior points of the triangle and that on the calculation of $\omega(T_-)$ they are not taken into account.

If the sector at the vertex of the angle $\alpha$ in the triangle $T$ is convex, then $\bar{\alpha} = \alpha$ and estimate (57) takes the form

$$\left\{ \begin{array}{l} \alpha - \alpha_0 \leq \bar{\alpha} - \alpha_0 \leq \omega^+(T_-), \\ \alpha_0 - \alpha \leq \omega^-(T_-) + \sum_{i=1}^{3} \tau_i^- \end{array} \right.$$ 

We turn to the proof of Theorem 10.

1. We excise the triangle $T$ from the manifold in question. If $T$ had interior tails, we cut $T$ along these tails. To the resulting polygon we paste along its contour the lateral surface of a right circular cylinder with a base having the same perimeter as that of the polygon. If $T$ has an exterior tail, we first paste along one of the sides containing the tail an arbitrarily narrow two-gon excised from the sphere by two meridians, whose lengths are taken equal to the lengths of the side of the triangle. Under all these pastings the angles $\bar{\alpha}, \bar{\alpha}, \alpha_0$ do not change or else change arbitrarily little.
After pasting, in the newly obtained manifold \( R \) the triangle \( T \) will be homeomorphic to the disc and absolutely convex. The angle between its sides corresponding to \( \alpha \) in the manifold \( R \) will be equal to \( \hat{\alpha} \).

2. By Theorem 5 of Chapter II, for \( T \) in the manifold \( R \) we have
\[
\hat{\alpha} - \alpha_0 \leq \nu_\alpha^+,
\]
where \( \nu_\alpha^+ \) is defined in terms of the excess of the triangles with vertex \( A \) excised by shortest arcs from \( T \) in \( R \). The excesses of these triangles are not decreased if they are measured with respect to their sector angles, and the latter excesses, from relation (55), do not exceed \( \omega^+(T_-) \). Therefore the inequality
\[
(59) \quad \hat{\alpha} - \alpha_0 \leq \omega^+(T_-).
\]

3. Now we make use of the fact that between the sides of the triangle \( T \) in \( R \) there exists an angle in the weak sense. Indeed, for the sides of \( T \) in \( R \) all the conditions of Theorem 13 of Chapter IV are satisfied.

Moreover, any two points on the sides \( AB, AC \) of the triangle \( T \) may be joined in \( R \) by a shortest arc closest to \( A \). These shortest arcs form a system of shortest arcs satisfying the conditions of Theorem 7 of Chapter II. Therefore for the angle in the weak sense we have the following estimate:
\[
\alpha_0 - \hat{\alpha} = \alpha_0 - \hat{\alpha}_{\zeta S} \leq \nu_{\hat{\alpha}}.
\]

The quantity \( \nu^- \) is defined in terms of the negative excesses of the triangles excised by the indicated shortest arcs. If the excess of such a triangle is negative, then each of its angles is less than \( \pi \), and since at every vertex there is a piece of the cylinder pasted from outside with an angle at that vertex equal to \( \pi \), then the angles of such triangles coincide with their sector angles. Now the excesses with respect to the sector angles may be estimated by the second of the estimates (55). Thus we find that
\[
(60) \quad \alpha_0 - \hat{\alpha} \leq \omega^-(T_-) + \sum_{i=1}^{3} \tau_i^-.
\]

4. It remains for us to prove the inequality
\[
(61) \quad \alpha_0 - \hat{\alpha} \leq \omega^+(T)
\]
for the case when \( \hat{\alpha} > \pi \) and therefore \( \hat{\alpha} \equiv \hat{\alpha} \).

First we shall formulate a lemma which will be useful also in Chapter VIII.
Lemma 10. Suppose that $T$ is a normal triangle in $R$ homeomorphic to the disc. We fix an arbitrarily small number $0 < \delta < 1/4$ and construct three isosceles triangles $t_1, t_2, t_3$ with angles at the bases equal to $\delta$ and lateral sides equal in length to half the sides of $T$. We are now going to carry out certain ever finer triangulations $Z_n$ of the triangle $T$. For each triangulation $Z_n$ we construct a polyhedral development $R'_n$ corresponding to $T$, and to it, along the boundary broken curves corresponding to $T$, we paste the lateral sides of the triangles $t_1, t_2, t_3$. Thus we obtain polyhedral developments $R_n$, as in Figure 86.

We assert that for any fixed $\delta > 0$ and for some form of sufficiently finely-constructed triangulation $Z_n$, the bases of the triangles $t_1, t_2, t_3$ will be shortest arcs in the development $R_n$.

Proof. We will construct the triangulations $Z_n$ in such a way that in accordance with Theorem 10 of Chapter III the metrics $\rho_n'$ of the developments $R'_n$ will converge as $n \to \infty$ to the original metric $\rho_T$ induced by excision of $T$ from $R$.

Now we suppose that for an infinite sequence of values of $n$ (we shall stick only to those $n$) the base $l$ of the triangle $t_i$ ($l = x \cos \delta$, where $x$ is the length of the side of $T$) is not a shortest arc in $2R_n$. Then for each $n$ the points $A$ and $B$, the endpoints of $l$, may be joined in $R_n$ by a shortest arc $l''$ shorter than $l$. The curve $l''$ may pass partly in $R'_n$, partly outside $R'_n$ in the pasted triangles $t_1, t_2, t_3$. Since $l''$ is a shortest arc and $\delta < \pi/4$, the curve $l''$ can go through each of the $t_1, t_2, t_3$ not more than once, “cutting off” an obtuse angle of the corresponding triangle.

Represent the length of $l''$ in the form of two terms $l'' = l_1'' + l_2''$, where $l_1''$ is the total length of the pieces of $l''$ passing inside $t_1, t_2, t_3$, and $l_2''$ is the total length of the pieces of $l''$ passing inside $R'_n$.

\[(62) \lim_{n \to \infty} l_2'' = 0.\]

Indeed, if each piece of $l_1''$, “cutting off” a vertex of $t_1, t_2, t_3$, is replaced by the corresponding circuit along the boundary pasted to $R'_n$ of $t_1, t_2, t_3$, then we obtain a length not larger than $l_1''/\cos \delta$. Along with the pieces $l_2''$ they form a length which for large $n$, in view of the convergence
\( \rho_n' \to \rho_T \), cannot be much shorter than \( x \). Therefore
\[
\frac{l_1^n}{\cos \delta} + l_2^n \geq x - \varepsilon_n, \quad \varepsilon_n \to +0.
\]

But by hypothesis
\[
l_1^n + l_2^n < x \cos \delta.
\]

Along with the preceding inequality this yields
\[
l_2^n < \frac{\varepsilon_n \cos \delta}{1 - \cos \delta}
\]
which proves equation (62).

The shortest arc \( l^n \), passing from \( A \) to \( B \) in the development \( R_n \), may arrive in various orders at three, two, or one of the triangles \( t_1, t_2, t_3 \). But from relation (62) and the fact that \( T \) is homeomorphic to a disc it follows that for sufficiently large \( n \) no possibilities occur except the following two: either \( l^n \) encounters only \( t_1 \), or \( l^n \) encounters first \( t_2 \) and then \( t_3 \). Other alternations of arrivals for large \( n \) contradict the smallness of \( l_2^n \).

Suppose that the shortest arc \( l^n \) arrives only at \( l_1 \) but does not coincide with \( l \). Then with the notation indicated in Figure 87 we have \( l^n = \overline{AA''} + A^nB^n + B^nB \). From (62) and the convergence \( \rho_n' \to \rho_T \) it follows that as \( n \to \infty \) in \( T \) the points \( A^n \) and \( B^n \) converge to \( A \) and \( B \) respectively. In a small neighborhood of the point \( A \) the absolute curvature of \( T \) and therefore the polyhedral metric \( R_n \) become very small. Therefore there the ratio of the length of the shortest arc \( \overline{AA''} \) to the length of a polygonal curve of small rotation, such as the curve \( \overline{AA''} \) of the pasting of \( t_1 \) to \( R'_n \), turns out for large \( n \) to be larger than \( \cos \delta \), given only that \( A^n \cong A \). Analogously for the piece \( BB^n \). Therefore for large \( n \) we will have \( \overline{AA''} \cong AM^n, \quad A^nB^n \cong M^nN^n, \quad B^nB \cong N^nB \), from which \( l^n \geq AB = l \), which contradicts the hypothesis \( l^n < l \).

Now suppose that all the shortest arcs \( l^n \) encounter \( t_2 \) and \( t_3 \), as in Figure 88. This time, with the notations indicated in Figure 88, it follows that in \( T \) \( A^n \to A, \quad B^n \to B, \quad C_2^n \to C, \quad C_3^n \to C \). Therefore we conclude that in the initial \( T \) we had the relation \( AC + CB = AB \) i.e., \( AC + CB \) constituted one shortest arc. Therefore in \( T \) at the vertex \( C \) the sector angle is not less than \( \pi \). For sufficiently
large \( n \) in the development \( R'_n \) the endpoints of the shortest arc \( C^n_1C^n_3 \), if they do not both merge with \( C \), will be joined also by the polygonal curve \( C^n_2CC^n_3 \), while the positive part of its rotation and the absolute curvature of the polyhedral metric between them will be very small. Therefore the ratio \( \frac{C^n_2C^n_3}{C^n_2CC^n_3} \) for sufficiently large \( n \) will not be less than \( \cos \delta \). Thus, for sufficiently large \( n \) we get \( \bar{A}A^n \geq AM_2, A^nC^n_2 \geq M_2N_2, C^n_2C^n_3 \geq N^n_2CN^n_3, C^n_3B^n \geq N^n_3M^n_3, B^nB \geq M^n_2B \). Therefore we get \( l^n \geq AC + CB = l \), which contradicts the hypothesis \( l^n < l \). Lemma 10 is proved.

Now we shall complete the proof of Theorem 10, i.e., prove inequality (61) for the case when \( \alpha > \pi \). We proceed as follows. We decompose the sectors at the vertices of \( T \) by shortest arcs into sectors less than \( \pi \). For these the angles \( \beta \) and \( \bar{\beta} \) will coincide. Moreover, we single out inside and on the sides of \( T \) in \( R \) almost all points at which the complete angle \( \theta > 2\pi \) and where in connection with this it is possible to have a difference between the angles \( \beta \) and \( \bar{\beta} \). More precisely, we select such a finite number of such points that for any choice of the remaining points the condition \( \sum(\theta - 2\pi) < \varepsilon \) will be satisfied. The sectors around the points thus singled out will be also subdivided by shortest arcs into angles less than \( \pi \). Finally, we decompose the whole triangle \( T \) into convex triangles \( t \), including among the curves of the subdivision the initial pieces of all the shortest arcs just drawn.

In rectifying the triangles thus obtained we may use the relation (59) already proved:

\[
\bar{\beta} - \beta_0 \leq \omega^+(t).
\]

But for the majority of angles we will have \( \bar{\beta} = \bar{\beta} \). In any case for any group of such angles

\[
\sum \bar{\beta} - \sum \beta < \varepsilon.
\]

From the rectified triangles we put together a polyhedral development
$R'$ with the preceding rule of overlapping. Along the polygonal curves into which the sides of $T$ were converted in $R'$, we paste the earlier isosceles triangles with very small angles $\delta$ at the bases, as in Figure 86. From Lemma 10 we may suppose that the development $R'$ was constructed with respect to a subdivision such that in the resulting development $R''$ the bases of the pasted isosceles triangles will be shortest arcs. At the same time the sector angles at the vertices of $T$ and the positive curvature of the polyhedral metric change by very little.

To the resulting metric and the triangle $T$ obtained above, it remains to apply the special case of formula (3) from Theorem 1 of subsection 4 of Chapter IV on the variation of the angle $\gamma$ in a polyhedral metric, in order to obtain an estimate of the deviation of the sector angle in $T$ in the metric of $R''$ from the angle in the plane triangle with the same length of sides, in terms of $\omega^\pm(T)$. Along with the arbitrary smallness of the $\delta$ and $\epsilon$ we are using, this leads us to estimate (61).

Theorem 10 is completely proved. We note that the estimates obtained in it can in general not be improved.

For abbreviation let us set,

$$\bar{\omega}^-(T) = \omega^-(T_\pm) \sum_{i=1}^{3} \tau_i^-,$$

where the $\tau_i^-$ are the negative parts of the rotations of the sides of the triangle $T$.

**Theorem 11.** For any combination of one, two, or three angles of the triangle $T$ the following estimates hold:

(63)  $$-\bar{\omega}^-(T) \leq \ddot{\alpha} - \alpha^0 \leq \omega^+(T^-),$$

(64)  $$-\bar{\omega}^-(T) \leq (\ddot{\alpha} - \alpha^0) + (\ddot{\beta} - \beta^0) \leq \omega^+(T^-),$$

(65)  $$-\bar{\omega}^-(T) \leq (\ddot{\alpha} - \alpha^0) + (\ddot{\beta} - \beta^0) + (\ddot{\gamma} - \gamma^0) \leq \omega^+(T^-).$$

**Proof.** Inequality (63) was proved in Theorem 10. Inequality (65) follows from the equation

(66)  $$\delta(T) = (\ddot{\alpha} - \alpha^0) + (\ddot{\beta} - \beta^0) + (\ddot{\gamma} - \gamma^0) = \omega^+(T^-) - \bar{\omega}^-(T),$$

proved in Theorem 9.

Inequality (64) follows from equation (66) and inequality (63), applied to the angle $\ddot{\gamma}$,

$$(\ddot{\alpha} - \alpha^0) + (\ddot{\beta} - \beta^0) = \ddot{\delta}(T) - (\ddot{\gamma} - \gamma^0) \leq \omega^+(T^-) - \bar{\omega}^-(T) + \bar{\omega}^-(T) = \omega^+(T^-),$$

$$(\ddot{\alpha} - \alpha^0) + (\ddot{\beta} - \beta^0) = \ddot{\delta}(T) - (\ddot{\gamma} - \gamma^0) \geq \omega^+(T^-) - \bar{\omega}^-(T) - \omega^+(T^-) = -\bar{\omega}^-(T).$$
Chapter VII

Converging Metrics

1. Convergence of metrics. In the study of figures in metric spaces it is useful to approximate them by other figures lying in the same space or in spaces approximating the given space. The approximation of figures and spaces may be given various meanings. In this section we give some possible definitions, enumerating them for convenience. The exposition refers to two-dimensional manifolds of bounded curvature.

1. Convergence of figures in metric space.

Definition 1 (Convergence of curves). The arcs \( L^n \) in the space \( R \) with the metric \( \rho \) converge in this space to the arc \( L \) if there exist parametrizations \( L^n = X^n(t), \ L = X(t), \ 0 \leq t \leq 1 \), of these arcs, under which for any \( \varepsilon > 0 \) for \( n \geq N(\varepsilon) \) and for all \( t \in [0,1] \) the inequality

\[
\rho(X^n(t), X(t)) < \varepsilon
\]

is satisfied.

Definition 2 (Convergence of polygons). Suppose that in a metrized two-dimensional manifold there are situated polygons \( P, P^n (n = 1, 2, \cdots) \), i.e., compact connected two-dimensional manifolds with edges, bounded by a finite number of simple closed polygonal curves. We shall say that the polygons \( P^n \) converge to \( P \) along with vertices and sides if beginning with some \( n \) the following conditions are satisfied:

1) The polygons \( P^n \) and \( P \) have the same number of vertices \( A^n, \ A \), and these may be so associated that the vertices \( A^n \) converge to the corresponding vertices \( A \), the sides of \( P^n \), \( P \) join corresponding pairs of vertices, and the sides of \( P^n \) converge in the sense of Definition 1 to the corresponding sides of \( P \).

2) There exists at least one interior point of the polygon \( P \) which is also interior for all \( P^n \).

Under the conditions of Definition 2 it is not difficult to show that every compact set \( K \subset P \) lies inside all the \( P^n \) for \( n \geq n(K) \).

For nonconnected polygons one may speak of the corresponding convergence of their separate components.

Definition 2 may also be used in the case when the “sides” of \( P^n, P \) are
Below we shall have in view the following definition of a sector. Suppose that from the point $O$ in a two-dimensional manifold of bounded curvature there issue two simple curves $L$ and $M$ not having common points other than $O$. Suppose that $U$ is a neighborhood of $O$ homeomorphic to the closed disc and sufficiently small so that $L$ and $M$ depart from the limits of $U$. The pieces of the curves $L$ and $M$ up to the first intersection with the boundary of $U$ divide the neighborhood $U$ into two pieces $U', U''$, which we shall call bounded sectors. The pieces of the curves $L$ and $M$ themselves are counted in both bounded sectors. Bounded sectors $U_1', U_1''$ obtained for different neighborhoods $U_1, U_2$ are regarded as equivalent if there exists a bounded sector $U_3' \subset U_1', U_3'' \subset U_2'$. The equivalence of bounded sectors is reflexive, symmetric, and transitive. An equivalence class of bounded sectors will be called a sector, the point $O$ its vertex, and $L$ and $M$ its sides.

If the curves $L$ and $M$ coincide on the portion adjacent to $O$, and then have no further common points, we also say that they form two sectors, one of which is degenerate. The bounded sectors for it are the pieces $L = M$, and for the complementary one nondegenerate sectors. Both sectors cannot be degenerate at the same time. In this definition, in distinction from subsection 7 of Chapter IV, we exclude the case of multiple contact and divergence of the sides of the sector.

A concrete sector may be indicated by giving a bounded sector. One may indicate a definite one of two nondegenerate sectors by giving the orientation of a neighborhood of the point $O$ and distinguishing in the order of circuit the sectors from $L$ to $M$ and from $M$ to $L$. A nondegenerate sector may be characterized by “a curve passing in it” (i.e., a curve issuing from $O$ and passing, on its initial portion, in bounded sectors of one class) or a sequence of points tending to $O$ “in this sector”.

**Definition 3 (Convergence of sectors).** Suppose that in a two-dimensional manifold of bounded curvature there are given a sector $V$ with vertex $O$ and sides $L, M$ and a sequence of sectors $V_n$ with vertices $O_n$ and sides $L_n, M_n$. We shall say that the sectors $V_n$ converge to the sector $V$ if the following conditions are satisfied: 1) $O_n \rightarrow O$; 2) there exists a neighborhood $U$ of the point $O$ homeomorphic to the disc such that if we restrict consideration to the pieces of the curves from $O, O_n$ to the boundary of $U$, then $L_n \rightarrow L$, $M_n \rightarrow M$ in the sense of Definition 1; 3) if
the sector \( V \) is nondegenerate, then there exists at least one point \( A \) lying inside a bounded sector \( U' \) belonging to \( V \) which for \( n \geq N(U, A) \) lies inside all the \( U'_n \subset V_n \). But if the sector \( V \) is degenerate the analogous requirement is imposed on the supplementary sectors \( U'', U''' \).

Under the conditions of Definition 3, for a nondegenerate \( V \) not only the separate point \( A \), but also each compact \( K \subset U' \), for \( n \geq N(U', K) \), lies inside all the \( U'_n \).

Most of all we shall be dealing with sectors bounded by shortest arcs.

2. Convergence of spaces.

Definition 4 (Uniform convergence of metrics). The following definition differs from the definition of subsection 8 of Chapter I in that the mappings \( \phi_n \) are now required to be homeomorphisms. We shall say that the metric spaces \( R_n \) uniformly converge to the metric space \( R \), or that the metrics \( \rho_n \) uniformly converge to the metric \( \rho \), if there exist homeomorphic mappings \( \phi_n \) of the spaces \( R_n \) onto \( R \) such that for any \( \varepsilon > 0 \) for \( n \geq N(\varepsilon) \) and any \( X, Y \in R \)

\[
|\rho(X, Y) - \rho(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \varepsilon.
\]

As a rule we shall use Definition 4 in considering closed two-dimensional manifolds of bounded curvature and compact manifolds with edge, in particular in the majority of local arguments. For noncompact manifolds in the large we introduce a somewhat different definition.

Definition 5 (Local uniform convergence of metrics). Suppose that \( R_n, R \) are metrized two-dimensional manifolds. We shall say that \( R_n \) converges locally uniformly to \( R \) if there is in \( R \) a sequence of open sets \( G'_n \) with compact closures \( \overline{G}'_n \), where \( \overline{G}'_n \) are closed manifolds or compact manifolds with an edge while

\[
G'_1 \subset \overline{G}'_1 \subset G'_2 \subset \overline{G}'_2 \subset \cdots \subset \overline{G}'_n \subset G'_{n+1} \subset \cdots \subset R = \bigcup_{n=1}^{\infty} G_n,
\]

and there are given homeomorphic mappings \( \phi_n \) of the pieces \( \overline{G}_n \subset R_n \) onto \( \overline{G}'_n \subset R \) which satisfy the following requirements:

1) for any \( \varepsilon > 0 \) and any compact \( K \subset R, K \subset G' \), is satisfied for \( n \geq N(\varepsilon, K) \), and (1) holds for all \( X, Y \in K \);

2) for any \( \varepsilon, A > 0 \) and \( X_0 \in R_n \) for \( n \geq N(X_0, A, \varepsilon) \) and for all \( X \in R_n \) satisfying the condition \( \rho_n(X, \phi_n^{-1}(X_0)) \leq A \), the inequality \( \rho_n(X, G_n) \leq \varepsilon \) is satisfied.

Definition 4 may be considered as a special case of Definition 5 for \( G_1 = G_2 = \cdots = R_n \).
3. Convergence of figures in converging spaces. Suppose that $R_n$ and $R$ are metrized manifolds and $R_n \to R$ in the sense of Definition 5. We introduce the following definitions.

1'. The arcs $L_n$ lying in $R_n$ converge to the arc $L$ lying in $R$ if for $n \geq N$ the arcs $L_n$ lie in the regions of definition of the $\phi_n$ and their images $L'_n = \phi_n(L_n)$ converge in $L$ in the sense of Definition 1.

2'. The polygons $P_n \subset R_n$ converge along with their vertices and sides to the polygon $P \subset R$, if for $n \geq N$ all the $P_n$ lie in the regions of definition of the $\phi_n$ and the figures $P'_n = \phi_n(P_n)$ converge in $R$ to $P$ in the sense of Definition 2, taking of course account of the fact that the sides of $P'_n$ may not be shortest arcs in $R$.

3'. The sectors $V_n$ of $R_n$ converge to the sector $V$ in $R$ if for $n \geq N$ their vertices $O_n$ lie inside the regions of definition of the $\phi_n$ and the images $V'_n$ of the sectors $V_n$, which are sectors in $R$, converge in $R$ to $V$ in the sense of Definition 3.

2. Converging curves and polygons.

4. Lengths of converging curves.

Theorem 1. Suppose that $R_n, R$ are metric spaces, with the $R_n$ converging uniformly to $R$ (or else $R_n, R$ are metrized two-dimensional manifolds and the $R_n$ locally uniformly converge to $R$), and suppose that the rectifiable curves $L_n \subset R_n$ converge to the curve $L \subset R$ in the sense of Definition 1'. Then

$$s(L) \leq \liminf_{n \to \infty} d(L_n),$$

where the length $s(L)$ is measured in $R$ and the lengths $s(L_n)$ in $R_n$.

Proof. Suppose that $\rho_n$ and $\rho$ are the metrics of the spaces $R_n$ and $R$, and that $\phi_n$ are homeomorphisms of regions of the spaces $R_n$ containing the $L_n$ onto a region $R$ containing $L$, under which in some compact neighborhood $G$ of the curve $L$ one has uniform convergence $\rho_n \to \rho$ and convergence $L'_n = \phi_n(L_n) \to L$. The points of the curves $L_n, L'_n, L$ corresponding with respect to the parameter under this convergence will be denoted by $X_n, Y_n, X$.

We select on $L$ an arbitrary sequence of points $X^i (i = 0, 1, \cdots, n)$. Then we have:

$$\sum_{i=1}^{m} \rho(X^{i-1}, X^i) \leq \sum_{i=1}^{m} [\rho(X^{i-1}, Y^{i-1}_n)] + \rho(Y^{i-1}_n, Y^i_n) + \rho(Y^i_n, X^i).$$

Because of the convergence $\rho_n \to \rho$ for $n \geq N_1(\varepsilon)$ this sum
\[ \leq \sum_{i=1}^{m} \rho(Y_{n}^{-1}, Y_n) + 2\varepsilon. \]

Because of the uniform convergence \( \rho_n \to \rho \) this sum
\[ \leq \sum_{i=1}^{m} \rho_n(X_{n}^{-1}, X_n) + 3\varepsilon \leq s(L_n) + 3\varepsilon. \]

Because of the arbitrariness of the original system of points \( X^i \) it therefore follows that for \( n \geq N(\varepsilon) \)
\[ s(L) \leq s(L_n) + 3\varepsilon. \]

Passing to the lower limit with respect to \( n \) and using the arbitrary smallness of \( \varepsilon > 0 \), we therefore obtain inequality (2).

**Theorem 2.** Suppose that \( R_n \) and \( R \) are metric spaces, while \( R_n \) uniformly converges to \( R \) (or \( R_n \) and \( R \) are metrized two-dimensional manifolds and \( R_n \) locally uniformly converges to \( R \)), and suppose that the lengths of certain rectifiable curves \( L_n \subset R_n \) are bounded uniformly by the number \( S \). Suppose moreover that the \( L_n \) lie in regions \( G_n \subset R_n \), mapped into \( R \) by those homeomorphisms \( \phi_n \) under which the convergence of \( R_n \) to \( R \) takes place, while \( \phi_n(G_n) \subset K \), where \( K \) is a compact region in \( R \).

Then one may select a subsequence from the system of curves \( L_n \) which converges under the mappings \( \phi_n \) to some curve \( L \subset R \).

**Proof.** 1. We select from all \( L_n = X_n(t) \) as a parameter \( 0 \leq t \leq 1 \) the relative arc length, and we choose a subsequence of curves for which the points \( Y_n(t) = \phi_n(X_n(t)) \) converge in \( R \) as \( n \to \infty \) to some point \( X(t) \) for each rational \( t \in [0, 1] \). This is possible in view of the countability of rational \( t \) and the compactness of \( K \). In what follows we shall consider only the \( L_n \) from this subsequence of curves.

2. Taking an arbitrary \( \varepsilon > 0 \), we choose rational \( t_i = i/m \), where \( i = 1, 2, \ldots, m \) and \( m \geq S/2\varepsilon \).

Now suppose that \( t \) is any value of the parameter from \([0, 1]\). For some one of the \( t_i \) chosen above we will have \( |t_i - t| \leq \varepsilon/3 \). Then because of the uniform convergence of the metrics \( \rho_n \to \rho \) and the choice of the parameters, we will have for sufficiently large \( n \):
\[ \rho(Y_n(t_i), Y_n(t)) < \rho_n(X_n(t_i), X_n(t)) + \varepsilon \leq |t_i - t| S + \varepsilon < 2\varepsilon. \]

Therefore for \( n, m \geq N(\varepsilon) \), taking account of the convergence of the points \( Y_n(t_i) \), we obtain
\[ \rho(Y_n(t), Y_m(t)) \leq \rho(Y_n(t), Y_n(t_i)) + \rho(Y_n(t_i), Y_m(t_i)) + \rho(Y_m(t_i), Y_m(t)) \]
\[ < 2\varepsilon + \rho(Y_n(t_i), Y_m(t_i)) + 2\varepsilon < 5\varepsilon. \]
Therefore it follows that $Y_n(t)$ converges to some point $X(t)$ and this convergence is uniform for all $t$.

3. For the limiting points $X(t)$, for $n \geq N(\varepsilon, t_1, t_2)$ we have

$$\rho(X(t_1), X(t_2)) < \rho(Y_n(t_1), Y_n(t_2)) + 2\varepsilon$$

$$\leq \rho_n(X_n(t_1), X_n(t_2)) + 3\varepsilon$$

$$\leq |t_1 - t_2| S + 3\varepsilon,$$

and thus

$$\rho(X(t_1), X(t_2)) \leq |t_1 - t_2| S,$$

so that the points $X(t)$ depend continuously on $t$ in $R$ and form some curve $L$.

Because of the uniform convergence $Y_n(t) \to X(t)$ proved above we have $L'_n \to L$. Theorem 2 is proved.

5. Convergence of polygons.

**Theorem 3.** Suppose that $R_n$ and $R$ are two-dimensional manifolds of bounded curvature with metrics $\rho_n, \rho$, and $P_n, P$ polygons in them, with $R_n$ converging uniformly to $R$ and $P_n$ converging to $P$ along with vertices and sides (in the sense of Definition 2'). Suppose that $\phi_n$ are homeomorphisms under which the uniform convergence of metrics $\rho_n \to \rho$ and the convergence of polygons hold. Then beginning with some $n$ the homeomorphisms $\phi_n$ can be replaced by homeomorphisms $\phi_n$ defined in the same regions under which we will also have uniform convergence of the metrics $\rho_n \to \rho$, and the polygons $P_n$ will be exactly mapped onto the polygon $P$ with preservation of the relations of vertices and sides (more precisely, with the preservation of that relation between the boundaries of $P_n$ and $P$ under which the uniform convergence of the boundaries $\phi_n(P)$ to the boundary of $P$ took place).

**Proof.** For definiteness we restrict ourselves to the case when $R_n$ and $R$ are homeomorphic to the open disc and the polygons $P_n$, $P$ are each bounded by a simple closed polygon $\gamma_n$, $\gamma$. Moreover, for simplicity we shall suppose that the mapping $\phi$ is given on the entire space and that the uniform convergence of metrics holds also on the entire space. We give the construction of the homeomorphisms $\phi_n$ in the interior regions of $R_n$ and $R$; the continuation of these homeomorphisms to the portions of $R_n$ and $R$ exterior to $P_n$ and $P$ is constructed analogously.

1. Take an $\varepsilon_i > 0$. We may suppose that $2\varepsilon_i$ is significantly smaller than the length of the links of $\gamma$. Suppose that $9\varepsilon$ is the smallest of the distances
from a link of $\gamma$, shortened at the ends by $\varepsilon_1$, to the family of the remaining links of $\gamma$. (Of course $9\varepsilon \leq \varepsilon_1$.) We shall take $n$ so large that the metrics $\rho_n, \rho$ for corresponding pairs of points in $R_n, R$ differ by less than $\varepsilon$, and the points of $\gamma' = \phi_n(\gamma)$ and $\gamma$ corresponding under the parameter are distant in $R$ by less than $\varepsilon$. Then the pieces of the sides of $\gamma_n$ corresponding under the parameter to the links of $\gamma$ shortened by $\varepsilon_1$ are distant from the other links of $\gamma_n$ in $R_n$ by more than $6\varepsilon$.

2. We may enclose the curve $\gamma$ in $R$ by an open region $G$ homeomorphic to the plane annulus, with all the points of $G$ distant from $\gamma$ by less than $\varepsilon$. We shall suppose that $n$ is large and that $\gamma^\prime_n$ is so close to $\gamma$ that $\gamma^\prime_n \subset G$. Then the regions $\phi_n^{-1}(G)$ will encircle $\gamma_n$ in $R$ and all the points of these regions will be distant from $\gamma_n$ by less than $3\varepsilon$.

3. From Lemma 2 of Chapter VI, the curve $\gamma$ may be approximated by a simple broken curve $l$ lying in $G$ which, with respect to the parameter, will uniformly approximate $\gamma$ with accuracy $\varepsilon$ both in the position of corresponding points and in the lengths of corresponding pieces. The preimage $\phi_n^{-1}(l) = l_n$ will be a simple curve (but no longer a broken curve) in $R_n$.

4. Mark off on $\gamma$ a finite number of points $A^i$, of which the end ones on each link are distant by $\varepsilon_1$ from the vertices, and the remainder are arranged on each side so that the distance between them is not less than $9\varepsilon$ and not larger than $18\varepsilon$. The points $A^i_n$ on $\gamma_n$ corresponding to them with respect to the parameter will be distant from one another at distances differing from theirs by not more than $\pm 3\varepsilon$.

5. We join each point $A^i_n$ by a shortest path $A^i_nB^i_n$ to the curve $l_n$. Analogously we join $A^i$ by a shortest path $A^iC^i$ to the curve $l$. Since the points $A^i_n$ are far from one another and close to $l_n$, these shortest arcs will not intersect one another. In view of 1 above they cannot intersect the links of $\gamma_n$ (or of $\gamma$) other than the link on which $A^i_n$ (or $A^i$) lies.

6. Replacing, if necessary, pieces of the shortest arc $A^i_nB^i_n$ by a piece of a link of $\gamma_n$, we may suppose that this shortest arc departs from the boundary $\gamma_n$ at $A^i_n$ or close to this point. In the latter case we replace the piece of the shortest arc adjacent to $\gamma_n$ by a piece of a polygonal curve of nearby length, which, with the exclusion of the endpoint $A^i_n$, lies inside $P$. Analogously for $A^iC^i$.

7. The endpoints $B^i_n, C^i$ need not correspond to one another under the homeomorphisms $\phi_n$. Suppose that $\tilde{B}^i_n = \phi_n(\dot{B}^i_n)$. The points $\tilde{B}^i_n$ lie on $l$ near to $C^i$. We replace the shortest arcs $A^iC^i$ by the polygonal curves $A^i\tilde{B}^i_n$
close in length and also lying in the strip between \( \gamma \) and \( l \).

8. Now we define a homeomorphism \( \phi_n \) in the following way. Suppose that in the region bounded by the curve \( l \), the homeomorphism \( \phi_n \) coincides with \( \phi_n \). The remaining piece of \( P_n \) is decomposed into a region \( A_n^t A_n^{t+1} B_n^{t+1} B_n^t \) homeomorphic to the disc. We map each such region homeomorphically onto the region \( A^t A^{t+1} B_n^{t+1} \) corresponding to it, with the preservation of the following relations on the boundary: the pieces \( B_n^{t+1} B_n^t \) the same as under the mapping \( \phi_n \); the pieces \( A_n^t A_n^{t+1} \) the same as in the correspondence of the parameters \( \gamma_n \leftrightarrow \gamma \), and the pieces \( B_n^t A_n^t \) according to the correspondence of the relative lengths.

The homeomorphism just constructed maps \( P_n \) onto \( P \) with the needed relations of the boundaries. The metrics \( \rho_n \) and \( \rho \) can differ under this relation by not more than \( C\varepsilon \). Beginning with some \( n \), we pass to the analogous construction of \( \phi_n \) with a new \( \varepsilon_2 \) in place of \( \varepsilon_1 \) and so forth, and thus construct the needed sequence of homeomorphisms. This proves Theorem 3.

6. **Convergence of the induced metrics.** If the set \( M \) in the metric space \( R \) is metrically connected (i.e., any two points \( X, Y \in M \) may be joined by a curve \( X\tilde{Y} \) of finite length lying in \( M \)), then the selection of \( M \) from \( R \) induces on \( M \) the metric

\[
\rho_M(X, Y) = \inf_{XY \in M} [s(X\tilde{Y})].
\]

A polygon in a two-dimensional manifold of bounded curvature is always metrically connected.

**Theorem 4.** Under the conditions of Theorem 3 the metrics \( \rho_{P_n} \), induced by the selection in \( R_n \) of the polygons \( P_n \), converge uniformly to the metric \( \rho_P \) induced by the selection of \( P \) from \( R \), this convergence being understood in the following two senses:

a) for any \( \varepsilon > 0 \) and compact \( K \subset P \), for \( n \geq N(\varepsilon, K) \) any pair of points \( X, Y \in K \) falls within the region of values of the homeomorphism \( \phi_n \) and

\[
|\rho_P(X, Y) - \rho_{P_n}(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \varepsilon;
\]

b) if \( \phi_n \) are taken to be those homeomorphisms whose existence was asserted in Theorem 3, then for \( n \geq N(\varepsilon) \) and for any \( X, Y \in P \)

\[
|\rho_P(X, Y) - \rho_{P_n}(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \varepsilon.
\]

**Proof.** 1. We first prove that for any fixed pair of points \( A, B \) lying within \( P \), and their preimages \( A_n = \phi_n^{-1}(A), B_n = \phi_n^{-1}(B) \), which for suf-
sufficiently large $n$ fall inside $P_n$, we have the convergence
\begin{equation}
\rho_p(A, B) = \lim_{n \to \infty} \rho_{P_n}(A_n, B_n).
\end{equation}

2. We join each pair $A_n$ and $B_n$ in $P_n$ by a shortest arc $\overline{A_nB_n}$. Because of the uniform convergence $\rho_n \to \rho$ and the convergence $P_n \to P$ the diameters and perimeters of $P_n$ are uniformly bounded, and therefore bounded in length $s_n(\overline{A_nB_n})$. From a sequence $\overline{A_nB_n}$, for which $s_n(\overline{A_nB_n}) \to \liminf_{n \to \infty} s_n(\overline{A_nB_n})$ we may by Theorem 2 select a subsequence converging to some curve $\overline{AB}$. Then we obtain
\begin{equation}
\rho_p(A, B) \leq s(\overline{AB}) \leq \liminf_{n \to \infty} s_n(\overline{A_nB_n}) = \liminf_{n \to \infty} \rho_{P_n}(A_n, B_n).
\end{equation}

3. Now join $A$ and $B$ by a shortest arc $\overline{AB}$ in $P$. It may be drawn so that this curve either does not intersect, or has one point in common with (or one common segment in common with) each of the links of the boundary $\gamma$ of the polygon $P$. Suppose that $m$ of the vertices of $P$ fall on $\overline{AB}$. Choose an arbitrary $\epsilon > 0$ and choose $\epsilon_1 > 0$ so small that $\epsilon_1 \leq \epsilon/12m$, and, beginning with some $n$, each of the vertices of $P_n$ is distant in $R_n$ by more than $6\epsilon_1$ from the boundary $\gamma_n$ of the polygon $P_n$, the sides issuing from this vertex being substracted. In what follows we shall consider only such $n$.

4. Subdivide $\overline{AB}$ by points $A'$ into a finite number of pieces $L_i = A'^iA'^{i+1}$ which are absolutely shortest arcs in $R$ of the following four types:

I) $L_i$ lies inside $P$;

II) $L_i$ lies inside $P$ with the deletion of one endpoint, which lies inside a side of $P$;

III) $L_i$ lies entirely inside one side of $P$;

IV) $L_i$ arrives at a vertex of $P$ at one of the endpoints.

There will not be more than $2m$ pieces of the fourth type. Adding if necessary division points, we may assume that each of the pieces of the fourth type is shorter than $\epsilon_1$.

5. For each point $A_i'$ falling on $\gamma$ there is a point $A_i' \in \gamma_n$ corresponding with respect to the parameter on the boundary. But if $A_i'$ lies inside $P_n$, we put it into correspondence with $A_i' = \phi_n^{-1}(A_i')$. Beginning with some $n$, all $A_i' \in P_n$. We shall moreover suppose that $n$ is so large that $|\rho_n - \rho| < \epsilon_1$ and $\rho(\gamma, \phi_n(\gamma_n)) < \epsilon_1$. Then for all points $A_i$
\begin{equation}
\rho_n(A_i', \phi_n^{-1}(A_i')) \leq \rho_n(A_i', A_i) + \epsilon_1 \leq 2\epsilon_1.
\end{equation}

6. For pieces of the fourth type we then have:
\begin{equation}
\rho_n(A_i', A_i'^{i+1}) \leq \rho_n(\phi_n^{-1}(A_i'), \phi_n^{-1}(A_i'^{i+1})) + 4\epsilon_1 \leq \rho(A_i', A_i'^{i+1}) + 5\epsilon_1 \leq 6\epsilon_1.
\end{equation}
One of the points $A^i_n, A^{i+1}_n$ is a vertex of $P_n$. It is distant from the sides not issuing from it by more than $6\varepsilon_1$, so that the absolute shortest arc $A^i_nA^{i+1}_n$ in $R_n$ may be regarded as not leaving $P_n$. Hence

$$\rho_{P_n}(A^i_n, A^{i+1}_n) = \rho_n(A^i_n, A^{i+1}_n) \leq 6\varepsilon_1. \quad (7)$$

7. We shall show that on each of the pieces of the first three types

$$\rho(A^i, A^{i+1}) \geq \limsup_{n \to \infty} \rho_{P_n}(A^i_n, A^{i+1}_n). \quad (8)$$

If $A^iA^{i+1}$ is a piece of the first type, i.e., $A^i_nA^{i+1}_n$ lies inside $P$, then this piece may be subdivided into segments $a^k\bar{a}^{k+1}$ small in comparison with their distances to $\gamma$ and so small that for sufficiently large $n$ their preimages $a^*_n, a^{*+1}_n$ lie inside $P_n$ substantially closer to one another than to $\gamma_n$. Then the absolute shortest arc $a^*_n a^{*+1}_n$ passes inside $P_n$. Therefore for sufficiently large $n$

$$\rho_{P_n}(A^i_n, A^{i+1}_n) \leq \sum_k \rho_{P_n}(a^*_n, a^{*+1}_n) = \sum_k \rho_n(a^*_n, a^{*+1}_n) \rightarrow \sum_{n \to \infty} \rho(a^k, a^{k+1}) = \rho(A^i, A^{i+1}), \quad (9)$$

from which inequality (8) follows.

If $A^iA^{i+1}$ is a piece of the second or third type, i.e., $A^i_nA^{i+1}_n$ has one endpoint in common with or entirely lies inside a side of $P$, then $A^iA^{i+1}$ may be subdivided into pieces $a^k\bar{a}^{k+1}$ small in comparison with their distance to the other sides of $P$. In this case one may mark off in $P_n$ points $a^*_n$ for which $\lim_{n \to \infty} \phi_n(a^*_n) = a^k$. For sufficiently large $n$ the absolute shortest arcs $a^*_n a^{*+1}_n$ in $R_n$ cannot intersect any side of $P_n$ other than the one corresponding side, and they may be regarded as passing in $P_n$.

8. Thus we have

$$\rho_P(A, B) = \sum \rho_{P}(A^i, A^{i+1}) = \sum \rho(A^i, A^{i+1}) \geq \sum' \rho(A^i, A^{i+1}),$$

where in $\sum'$ we have admitted all the pieces of the fourth type. Taking into account (8), we may for large $n$ prolong this inequality

$$\geq \sum' \rho_{P_n}(A^i_n, A^{i+1}_n) - \varepsilon;$$

by adding terms corresponding to pieces of the fourth type, taking account of inequality (7) we obtain

$$\geq \sum \rho_{P_n}(A^i_n, A^{i+1}_n) - \varepsilon - 6\varepsilon_1 2m \geq \rho_{P_n}(A_nB_n) - 2\varepsilon.$$

Therefore, in view of the arbitrary smallness of $\varepsilon$ it follows that
\[ \rho_p(A, B) \geq \limsup_{n \to \infty} \rho_{p_n}(A_n, B_n) \]

which along with inequality (6) proves (5).

9. We have proved that the convergence (5) holds for each fixed pair of points \( A, B \in P \). We choose an arbitrary compact \( K \subset P \). For \( n \) sufficiently large, \( \psi_n(P_n) \supseteq K \). We shall show that the convergence (5) is uniform for all \( A, B \in K \). Suppose the contrary. Then for some \( \varepsilon_0 > 0 \) there exist pairs of points \( X_i, Y_i \in K \) \((i = 1, 2, \cdots)\), for which

\[ |\rho_p(X_i, Y_i) - \rho_{p_n}(X_{n,i}, Y_{n,i})| > 5\varepsilon_0, \]

as \( n_i \to \infty \). Because of the compactness of \( K \) we may suppose that we have already selected a subsequence such that \( X_i \to X_0, Y_i \to Y_0 \).

10. Suppose that \( Z \in P \) and \( \rho(Z, \gamma) \geq 6\varepsilon > 0 \). We choose an \( \varepsilon \)-neighborhood \( U \) of the point \( Z \), homeomorphic to the disc. Then for any \( x, y \in U \) the shortest arc \( xy \) does not issue from \( P \) and therefore \( \rho_p(x, y) = \rho(x, y) \).

Enclose \( \gamma \) in \( R \) by a strip \( G \) all of whose points are distant from \( \gamma \) by less than \( \varepsilon \). Suppose that \( n \) is so large that \( |\rho_n - \rho| < \varepsilon \), \( \rho(\phi_n(\gamma_n), \gamma) < \varepsilon \) and \( \phi_n(\gamma_n) \) lies in \( G \). Then \( \rho(Z, \phi_n(\gamma_n)) > 5\varepsilon \).

The regions \( U_n = \phi_n^{-1}(U) \) lie in \( P_n \). For any \( x_n, y_n \in U_n \) we have

\[ \rho_n(x_n, y_n) \leq \rho(x, y) + \varepsilon \leq 3\varepsilon, \]

\[ \rho_n(x_n, \gamma_n) \geq \rho(x, \phi_n(\gamma_n)) - \varepsilon \geq \rho(Z, \phi_n(\gamma_n)) - 2\varepsilon > 3\varepsilon. \]

Therefore the shortest arc \( x_ny_n \) cannot touch \( \gamma_n \) and lies in \( P_n \), so that \( \rho_{p_n}(x_n, y_n) = \rho_n(x_n, y_n) \).

11. Surround the points \( X^0 \) and \( Y^0 \) by neighborhoods \( U \) and \( V \) as was done under point 10 for the point \( Z \), putting \( \varepsilon \leq \varepsilon_0 \). Then in \( U, V \) we will have \( \rho_p = \rho \) and for sufficiently large \( n \) within the limits of \( U_n, V_n \) we will have \( \rho_{p_n} = \rho_n \).

Now suppose that \( X^i \) and \( Y^i \) are so close to \( X^0 \) and \( Y^0 \) respectively and \( n \), so large that the \( X^i \) and \( Y^i \) lie in \( U \) and \( V \) respectively. Their preimages \( X_n^i \) and \( Y_n^i \) in \( R_n \) lie in \( U_n \) and \( V_n \) respectively and are distant from \( X_n^0 \) and \( Y_n^0 \) respectively by less than \( \varepsilon_0 \). Moreover, because of the convergence (5), for \( A = X^0, B = Y^0 \)

\[ |\rho_p(X^0, Y^0) - \rho_{p_n}(X_n^0, Y_n^0)| < \varepsilon_0. \]

Then we have:

\[ |\rho_p(X^i, Y^i) - \rho_{p_n}(X_n^i, Y_n^i)| \leq |\rho_p(X^0, Y^0) - \rho_{p_n}(X_n^0, Y_n^0)| \]
\[ + |\rho_p(X^i, X^0)| + |\rho_p(Y^i, Y^0)| \]
\[ + |\rho_{p_n}(X_n^i, X_n^0)| + |\rho_{p_n}(Y_n^i, Y_n^0)| \]
\[ < 5\varepsilon_0, \]
which contradicts 10 above.

Thus assertion a) of Theorem 4 is completely proved.

12. Assertion b) may be proved in the following way. Consider the construction of the homeomorphisms \( \phi_n \) in Theorem 3. We may verify that for each \( \varepsilon > 0 \) there exists a number \( N \) and a compact \( K \subset P \) such that for \( n \geq N \) and for any \( A, B \in P \) there exist \( A', B' \subset K \) distant from them in \( \rho_p \) by less than \( \varepsilon \), while the preimages \( \phi_n^{-1}(A), \phi_n^{-1}(B) \) will be distant from \( \phi_n^{-1}(A'), \phi_n^{-1}(B') \) in \( \rho_p \) by less than \( \varepsilon \).

Moreover we may suppose that for \( n \geq N \) the homeomorphisms \( \phi_n \) and \( \phi_n \) coincide on \( K \) and that the metrics \( \rho_p \) and \( \rho_p \) on \( K \) and \( \phi_n^{-1}(K) \), from the already proved a) of Theorem 4, differ by less than \( \varepsilon \). Then for any \( A, B \in P \) we will have

\[
|\rho_p(A, B) - \rho_p(\phi_n^{-1}(A), \phi_n^{-1}(B))| < |\rho_p(A', B') - \rho_p(\phi_n^{-1}(A'), \phi_n^{-1}(B'))| + 2\varepsilon < 3\varepsilon,
\]

which proves assertion b) of Theorem 4.

3. Charges and weak convergence. In this section we shall formulate the definitions and some known properties of charges and weak convergence of set functions, which will be needed in this chapter. The proofs of these results, along with other properties of charges and of weak convergence, may be found in a paper of A. D. Aleksandrov \[1\]. We should like to emphasize that the formulation presented below is related to fully normal\[2\] topological spaces \( R \), which are certainly metrized manifolds.

7. Charges.

DEFINITION. By a charge in a space \( R \) we mean a function \( \mu(M) \) defined on Borel sets \( M \subset R \) which satisfies the following three conditions:

\begin{enumerate}
  \item \( \mu(M_1 \cup M_2) = \mu(M_1) + \mu(M_2) \) if \( M_1 \cap M_2 = 0 \);
  \item \( |\mu(M)| \leq N < +\infty \);
  \item for each \( \varepsilon > 0 \) there exists a closed \( F \subset M \) for which \( |\mu(M) - \mu(F)| < \varepsilon \).
\end{enumerate}

LEMMA 1. A bounded completely additive function defined on a field of Borel sets in \( R \) is always a charge.

For example, if from a two-dimensional manifold of bounded curvature

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1 See also V. I. Glivenko \[42\] and Ju. V. Prohorov \[60\].

2 The space \( R \) is called normal if in it any two nonintersecting closed sets \( F \) are separable by nonintersecting open sets, and fully normal, if moreover each set \( F \) in it is a set of type \( G_\delta \).
we distinguish a region with a compact closure, then within the limits of this region the functions $\omega, \omega^+, \omega^-, \Omega$ are charges.

**Lemma 2.** For each charge $\mu$ there exist nonnegative charges $\mu^+$ and $\mu^-$, called the positive and negative parts of the charge $\mu$, for which the equations

\[
\begin{align*}
\mu^+(M) & \equiv \sup_{E \subseteq M} \{\mu(E)\} = \sup_{F \subseteq M} \{\mu(F)\}, \\
\mu^-(M) & \equiv \sup_{E \subseteq M} \{-\mu(E)\} = \sup_{F \subseteq M} \{-\mu(F)\}
\end{align*}
\]

hold, where $E$ are Borel sets and $F$ closed sets.

The identities (11) constitute definitions of $\mu^+$ and $\mu^-$. 

**Lemma 3.** The representation $\mu = \mu^+ - \mu^-$ is valid. It is minimal in the following sense: for any representation $\mu = \mu_1 - \mu_2$, where $\mu_1$ and $\mu_2$ are nonnegative charges, the inequality

\[
\mu^+(M) \leq \mu_1(M), \quad \mu^-(M) \leq \mu_2(M)
\]

holds.

**Definition.** The variation of the charge $\mu(M)$ is the charge

\[
|\mu|(M) = \mu^+(M) + \mu^-(M).
\]

**Lemma 4.** For each nonnegative charge $\mu$

\[
\mu(M) = \inf_{G \supseteq M} \{\mu(G)\},
\]

where $G$ are open sets.

**8. Weak convergence.**

**Definition.** We say that the sequence of charges $\mu_n$ converges weakly if for any bounded continuous function $f(x)$, $X \subseteq L$, the limit

\[
\lim_{n \to \infty} \int_R f(x) \mu_n(dE)
\]

exists. The integral is understood in the sense of Lebesgue-Stieltjes.

**Lemma 5.** If the charges $\mu_n$ converge weakly, then they converge to some uniquely defined charge $\mu$, i.e., for each continuous bounded function $f(x)$

\[
\lim_{n \to \infty} \int_R f(x) \mu_n(dE) = \int_R f(x) \mu(dE).
\]

In this case we write $\mu_n \rightarrow_w \mu$. 
Lemma 6. The charges of a weakly converging sequence are uniformly bounded, i.e., $|\mu_n(M)| \leq N < +\infty$ for all $M$ and $n$.

Lemma 7 (Kolmogorov's Test). For a sequence of charges $\mu_n$ to converge weakly to the charge $\mu$ it is necessary and sufficient that they should be bounded uniformly, and that for each open $G_0$ and closed $F_0 \subset G_0$ the condition

$$\lim_{n \to \infty} \inf_{F \subset G \subset G_0} |\mu(G) - \mu_n(G)| = 0$$

should be satisfied.

Lemma 8 (Weak Compactness). From a sequence of charges which are uniformly bounded and given on a compact space one may select a weakly converging subsequence.

Lemma 9. If $\mu_m \to \mu$, then for any open set $G$

$$\mu^p(G) \leq \liminf_{m \to \infty} \mu_m^p(G), \quad \mu^n(G) \leq \liminf_{m \to \infty} \mu_m^n(G).$$

Lemma 10. If the charges $\mu_n$, $\mu$ are nonnegative, then for weak convergence $\mu_n \to \mu$ it is necessary and sufficient that the two following conditions be observed:

1) in the space in the large

$$\mu(R) = \lim_{n \to \infty} \mu_n(R);$$

2) for each closed $F \subset R$

$$\mu(F) \geq \limsup_{n \to \infty} \mu_n(F)$$
or, which in this case is the same, for any open $G \subset R$

$$\mu(G) \leq \liminf_{n \to \infty} \mu_n(G).$$

Lemma 11. If the $\mu_n$ and $\mu$ are nonnegative, then for weak convergence $\mu_n \to \mu$ it is necessary and sufficient that the equation

$$\mu(G) = \lim_{n \to \infty} \mu_n(G)$$

hold for all $G$ for which $\mu(\bar{G}) = \mu(G)$, i.e., $\mu(Fr. G) = 0$. Here $\bar{G}$ is the closure in $R$.

Lemma 12. If the $\mu_n$ are completely additive and $\mu_n \to \mu$, then $\mu$ is also a completely additive charge.

9. Escaping loads.

Definition. A sequence of nonempty closed sets $F_n$ is said to be di-
vergent if these sets do not intersect one another and the sum of any collection (including infinite collections) of sets \( F_n \) is closed in \( R \). We shall say that in a system of charges there is an \textit{escaping load}, if there exists a diverging sequence of closed sets \( F_n \) and a number \( a \neq 0 \) such that for all \( n \)

\[
\frac{\mu_n(F_n)}{a} \geq 1.
\]

**Lemma 13.** In a weakly converging sequence of charges there is no escaping load.

**Lemma 14.** If in a sequence of charges \( \mu_n \) there is no escaping load, then for any fixed diverging sequence \( F_n \) and \( \varepsilon > 0 \), for \( n > N(\varepsilon) \)

\[
|\mu_n|(F_n) < \varepsilon.
\]

**Lemma 15.** In a locally compact \( R \), for the weak compactness of a system of charges \( \mu_n \) it is necessary and sufficient that the \( \mu_n \) should be uniformly bounded and that there should be no escaping load.

10. **Local weak convergence.**

**Definition.** A sequence of charges \( \mu_n \) converges locally weakly to the charge \( \mu \) if for any continuous function \( f(x) \) distinct from zero only on a set with compact closure

\[
\lim_{n \to \infty} \int_R f(x) \mu_n(dE) = \int_R f(x) \mu(dE).
\]

**Lemma 16 (Kolmogorov's test).** For the local weak convergence of \( \mu_n \) to \( \mu \) it is necessary and sufficient that in each region \( G_0 \) with compact closure \( \overline{G_0} \) the charges \( \mu_n \) and \( \mu \) should be bounded uniformly and that equation (15) should be satisfied for each \( F_0 \subset G_0 \).

**Lemma 17.** From local weak convergence and the absence of an escaping charge follows weak convergence on all of \( R \).

4. **Curvatures of converging metrics**

11. **Curvature as a charge.** In a two-dimensional manifold of bounded curvature with a compact closure the completely additive functions \( \omega, \omega^+, \omega^- \) and \( \Omega \) are bounded and are therefore charges. The last three of these are nonnegative charges.

**Theorem 5.** The positive and negative parts \( \omega^+, \omega^- \) of the curvature \( \omega \) coincide with the positive and negative parts of \( \omega \) in the sense of the theory of charges, in other words
\[
\begin{align*}
\omega^+(M) &= \omega^o(M) = \sup_{E \in M} \{ \omega(E) \}, \\
\omega^-(M) &= \omega^o(M) = \sup_{E \in M} \{-\omega(E)\}.
\end{align*}
\]

**Proof.** Since \( \omega = \omega^o - \omega^o \) and \( \omega = \omega^+ - \omega^- \), it follows from Lemma 3 that \( \omega^o \leq \omega^+ \), \( \omega^o \leq \omega^- \), so that it suffices to prove the converse inequalities. Because of the fact that relation (13) is valid for \( \mu = \omega^o, \omega^o, \omega^+, \omega^- \), it is sufficient to prove equation (20) for open \( M \).

Suppose that \( G \) is an open set with compact closure. For any \( \varepsilon > 0 \), from the definition of \( \omega^+(G) \), there exists in the set \( G \) a system of triangles homeomorphic to the disc \( \{ T_i \} \) with excesses \( \delta(T_i) \geq 0 \), for which

\[ \omega^+(G) < \sum_i \delta(T_i) + \varepsilon. \]

From Theorem 9 of Chapter VI, for a triangle \( T_i \) homeomorphic to the disc

\[ \delta(T_i) = \omega(T_{\omega-i}) + \sum_{k=1}^3 \tau_k, \]

where \( \tau_1, \tau_2, \tau_3 \) are the certainly nonpositive rotations of the open sides of \( T_i \) from the side of the interior region of \( T_i \).

From the additivity of \( \omega \) and the nonpositiveness of \( \tau_k \) we conclude that

\[ \omega^+(G) < \omega \left( \sum_i T_{\omega-i} \right) + \varepsilon, \]

so that in view of the arbitrariness of \( \varepsilon > 0 \) it follows that

\[ \omega^+(G) \leq \omega^o(G). \]

Analogously for each \( \varepsilon > 0 \), from the definition of \( \omega^-(G) \), there exists a system of reduced triangles \( T_i \) in the set \( G \) with excesses \( \delta(T_i) \leq 0 \) for which

\[ \omega^-(G) < -\sum_i \delta(T_i) + \varepsilon. \]

But this time there may be triangles of five types among the \( T_i \), as depicted in Figure 89.

![Figure 89.](image)

If in triangles of types two through four we reject "exterior tails", thus replacing the \( T_i \) by triangles homeomorphic to the disc, and drop
triangles of the fifth type altogether, but here also add \(-\sum \omega(A_j)\), to 
\(-\sum \delta(T_i)\), where the \(A_j\) are the points serving as the bases of the rejected
"tails", then we will have

\[-\sum \delta(T_i) \leq -\sum \delta(t_i) - \sum \omega(A_j).\]

For each \(t_i\) homeomorphic to the disc we have:

\[\bar{\delta}(t_i) = \omega(t_{i-1}) + \sum_{k=1}^{3} \tau_i^k,\]

where the \(\tau_i^k\) are the rotations of the sides of \(t_i\). Here, on each piece of
a shortest arc its rotations are nonpositive, and the sum of the right and
left rotations is the curvature of that portion of the shortest arc. Therefore
we may write

\[\omega^-(G) \leq -\omega\left(\sum t_{i-1} + \sum (\text{sides } t_i) + \sum A_j\right) + \varepsilon.\]

Because of the arbitrariness of \(\varepsilon > 0\) it therefore follows that

\[\omega^-(G) \leq \omega^*(G).\]

Theorem 5 is proved.

12. **Angles of sectors with a common vertex.** Suppose that in a neigh-
borhood \(U\) of the point \(O\), homeomorphic to the disc, there are given
metrics \(\rho_n\), uniformly converging to the metric \(\rho\), with the absolute cur-
vatures of all these metrics bounded uniformly by a small number \(\varepsilon\):

\[(21) \quad \Omega(U) < \varepsilon, \quad \Omega_n(U) < \varepsilon, \quad (0 < \varepsilon < \pi).\]

Suppose that we have drawn from the point \(O\) in the neighborhood \(U\)two shortest arcs \(L\) and \(M\) in the metric \(\rho\) and shortest arcs \(L_n, M_n\) in
the metrics \(\rho_n\), with each pair \(L, M; L_n, M_n\) of these shortest arcs having
no common points other than \(O\) or a common initial segment issuing
from \(O\). Suppose finally that in the metric \(\rho\) the curves \(L_n\) converge to
\(L\) and the \(M_n\) to \(M\).

Supposing that the neighborhood \(U\) is oriented, we may for each pair
of shortest arcs \(L_n, M_n\); \(L, M\) distinguish two sectors: the first in the order
of the circuit of the point \(O\) from \(L_n\) or \(L\) to \(M_n\) or \(M\), and the second
the complementary sector. The angles of the first sectors will be denoted
by the superscript 1, and those of the second sectors by the superscript
2. If one of the sectors \(L, M\) is degenerate, we take it to be the first one.

**Lemma 18.** With the conditions stated, i.e., given the uniform convergence
of metrics and shortest arcs and with condition (21) observed, for the angles
\( \alpha, \alpha_n \) between the shortest arcs and the angles \( \tilde{\alpha}^1, \tilde{\alpha}^2, \tilde{\alpha}_n, \tilde{\alpha}_n^2 \) of the sectors between them, the following inequalities are valid for sufficiently large \( n \):

\[
(22) \quad |\alpha - \alpha_n| < 5\varepsilon, \\
(23) \quad |\tilde{\alpha}^1 - \tilde{\alpha}_n^1| < 9\varepsilon, |\tilde{\alpha}^2 - \tilde{\alpha}_n^2| < 9\varepsilon.
\]

**Proof.** 1. Choose points \( X \) and \( Y \) distinct from \( O \) on \( L \) and \( M \), and on \( L_n \) and \( M_n \) points \( X_n \) and \( Y_n \) that correspond to them with respect to the parameter on the converging curves. We may choose the points \( X \) and \( Y \) so close to \( O \) that any shortest arcs joining points of the segments \( OX, OY \) in \( \rho \) or \( OX_n, OY_n \) in \( \rho_n \) will not leave \( U \).

Join \( X \) and \( Y \) in \( \rho \) and \( X_n \) and \( Y_n \) in \( \rho_n \) by shortest arcs, forming with \( L, M \) and \( L_n, M_n \) reduced triangles \( T, T_n \). These triangles are then developed on the plane, with \( \alpha^0 \) and \( \alpha_n^0 \) the angles of these plane triangles corresponding to the vertex \( O \). From Lemma 23 in subsection 12 of Chapter V we have

\[
(24) \quad |\alpha - \alpha^0| \leq 2\Omega(U), \quad |\alpha_n - \alpha_n^0| \leq 2\Omega_n(U).
\]

Because of the uniform convergence \( \rho_n \to \rho \) and \( X_n \to X, Y_n \to Y \), the lengths of the sides of \( T_n \) converge to the lengths of the sides of \( T \). Therefore for sufficiently large \( n \) we will have the inequality

\[
(25) \quad |\alpha^0 - \alpha_n^0| < \varepsilon.
\]

Then

\[
|\alpha - \alpha_n| \leq |\alpha - \alpha^0| + |\alpha^0 - \alpha_n^0| + |\alpha_n^0 - \alpha_n| \leq 2\Omega(U) + \varepsilon + 2\Omega_n(U) < 5\varepsilon,
\]

which proves inequality (22).

2. In order to prove inequality (23), we consider first the case when for the construction just realized

\[
(26) \quad 0 \leq \alpha^0 \leq \pi - 2\varepsilon.
\]

In this case the shortest arc \( XY \) does not pass through \( O \). Suppose for definiteness that it passes in the sector with angle \( \tilde{\alpha}^1 \) and forms with \( L \) and \( M \) a triangle \( T \). From Theorem 10 of Chapter VI we have

\[
|\tilde{\alpha}^1 - \alpha^0| \leq \Omega(T) \leq \Omega(U) < \varepsilon.
\]

In the case at hand there cannot exist another shortest arc \( \bar{XY} \) lying in the sector with the angle \( \tilde{\alpha}^2 \). For otherwise we would have also

\[
|\tilde{\alpha}^2 - \alpha^0| < \varepsilon,
\]

which would lead to the inequality

\[
\Omega(0) = |2\pi - (\tilde{\alpha}^1 + \tilde{\alpha}^2)| = |2(\pi - \alpha^0) - (\tilde{\alpha}^1 + \tilde{\alpha}^2 - 2\alpha^0)| \geq 4\varepsilon - 2\varepsilon = 2\varepsilon,
\]
in contradiction to the fact that \( \Omega(O) \leq \Omega(U) < \varepsilon \).

For sufficiently large \( n \), when condition (25) is satisfied, the angles \( \alpha_n^0 < \pi \) and the shortest arcs \( X_n Y_n \) in \( \rho_n \) also do not pass through \( O \). Moreover, beginning with some \( n \), all of these shortest arcs pass through the sectors with angles \( \bar{\alpha}_n^1 \). For otherwise we would have a subsequence of shortest arcs which would converge to a shortest arc \( \bar{XY} \) in the metric \( \rho \), lying in the sector with angle \( \bar{\alpha}^2 \). But as we have shown above such a shortest arc does not exist.

For the triangle \( T_n \) excised in \( \rho_n \) by the shortest arcs \( X_n Y_n \), we have

\[
|\bar{\alpha}_n^1 - \alpha_n^0| \leq \Omega_n(T_n) \leq \Omega_n(U) < \varepsilon.
\]

Finally

\[
|\bar{\alpha}^1 - \bar{\alpha}_n^1| \leq |\bar{\alpha}^1 - \alpha^0| + |\alpha^0 - \alpha_n^0| + |\alpha_n^0 - \bar{\alpha}_n^1| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,
\]

so that in the case at hand certainly the first of the inequalities (23) holds. The second inequality follows from the relations

\[
|\bar{\alpha}^2 - \bar{\alpha}_n^2| = |(\bar{\alpha}^1 + \bar{\alpha}^2 - 2\pi) - (\bar{\alpha}_n^1 + \bar{\alpha}_n^2 - 2\pi) - (\bar{\alpha}^1 - \bar{\alpha}_n^1)| < \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon.
\]

3. Now we shall prove inequality (23) in the case when

\[
(27) \quad \pi - 2\varepsilon < \alpha^0 \leq \pi.
\]

In this case, from inequality (24), \( \pi - 4\varepsilon < \alpha \), so that \( \pi - 4\varepsilon < \bar{\alpha}^1 \), \( \pi - 4\varepsilon < \bar{\alpha}^2 \). Along with relation \( \bar{\alpha}^1 + \bar{\alpha}^2 < 2\pi + \varepsilon \), resulting from the bound \( \Omega(O) = |\bar{\alpha}^1 + \bar{\alpha}^2 - 2\pi| \) on the curvature, this shows that

\[
(28) \quad \pi - 4\varepsilon < \bar{\alpha}^1 < \pi + 5\varepsilon, \quad \pi - 4\varepsilon < \bar{\alpha}^2 < \pi + 5\varepsilon.
\]

From the fact that \( \alpha_n^0 \to \alpha^0 \) as \( n \to \infty \) it follows that for sufficiently large \( n \) the condition

\[
\pi - 2\varepsilon < \alpha_n^0 \leq \pi,
\]

analogous to (27), holds.

From this, as above, it follows that

\[
(29) \quad \pi - 4\varepsilon < \bar{\alpha}_n^1 < \pi + 5\varepsilon, \quad \pi - 4\varepsilon < \bar{\alpha}_n^2 < \pi + 5\varepsilon.
\]

From relations (28) and (29) follow inequalities (23). The lemma is completely proved. Of course, the constants in inequalities (22) and (23) may be decreased.

13. Weak convergence of curvature.

Theorem 6. Suppose that on one and the same two-dimensional manifold \( R \) there are given metrics \( \rho_n, \rho \) of bounded curvatures \( \omega_n, \omega \), with the metrics
\( \rho_n \), converging uniformly to the metric \( \rho \) and the absolute curvatures \( \Omega_n \) of the metrics \( \rho_n \), bounded uniformly. Then the curvatures \( \omega_n \), given as set functions on Borel sets, converge locally weakly to \( \omega \), i.e., for every continuous function \( f(x) \), distinct from zero only on a set with compact closure, the equation

\[
\lim_{n \to \infty} \int \int f(x) \omega_n(dE) = \int \int f(x) \omega(dE)
\]

is valid.

**Proof.** 1. From Lemma 16, for the proof of Theorem 6 it is sufficient to verify that the Kolmogorov test is fulfilled, i.e., to prove that for each open \( G_0 \) with compact closure and each closed \( F_0 \subset G_0 \) the relation

\[
\lim \inf_{n \to \infty} \inf_{F_0 \subset G \subset G_0} |\omega(G) - \omega_n(G)| = 0
\]

is fulfilled.

Suppose that for some pair \( F_0 \subset G_0 \), where \( G_0 \) is compact, this last relation is not true, i.e., for some \( \varepsilon_0 > 0 \) and some subsequence \( n \to \infty \), to which we shall hold in what follows, keeping to the notation 1,2,\( \cdots \), \( n \), the condition

\[
(30) \quad \inf_{F_0 \subset G \subset G_0} |\omega(G) - \omega_n(G)| > \varepsilon_0
\]

is fulfilled.

2. From Lemma 7, on a compact set \( \overline{G}_0 \), from a uniformly bounded system \( \Omega_n \), it is possible to select a uniformly convergent subsequence. Keeping only to this subsequence and retaining for it the numeration 1,2,\( \cdots \), \( n \), we will suppose that \( \Omega_n \wedge \Omega' \). Here, from Lemma 12, the function \( \Omega' \) will in its turn be a completely additive nonnegative charge, defined on the Borel subsets of the set \( \overline{G}_0 \).

3. Select an open set \( G \) such that \( F_0 \subset G \subset \overline{G} \subset G_0 \) and \( \Omega'(\text{Fr. } G) = 0 \). Such sets \( G \) certainly exist. It suffices to consider various \( \xi \)-neighborhoods of the set \( F_0 \). For sufficiently small \( \xi \) they are contained in \( G_0 \), and their boundaries are distinct for distinct \( \xi \). Because of the complete additivity of \( \Omega' \) only for a finite number of values of \( \xi \) can the value of \( \Omega' \) on these boundaries differ from zero.

Take now an arbitrarily small \( \varepsilon > 0 \). We choose a closed set \( F \) such that \( F_0 \subset F \subset G \), \( \Omega'(G-F) < \varepsilon \), \( \Omega(G-F) < 2\varepsilon \). This is possible since \( \Omega' \) and \( \Omega \) are charges.

Under these conditions, for sufficiently large \( n \), using property (18), we will have
\[ \Omega_n(G - F) < \Omega'(G - F) + \varepsilon < 2\varepsilon. \]

4. From Theorem 9 of Chapter V,

\[ \omega(G) = \lim_{P \subseteq G, P \to G} \delta(P), \]

so that there exists a polygon \( P_\varepsilon \subseteq G \) such that for all polygons \( P \) with \( P_\varepsilon \subseteq P \subseteq G \)

\[ (31) \quad |\omega(G) - \delta(P)| < \varepsilon. \]

We shall moreover suppose \( P_\varepsilon \) chosen so that \( F \) is contained inside \( P_\varepsilon \), i.e., \( F \subseteq P_\varepsilon \).

5. Select a polygon \( P \) so that \( P_\varepsilon \subseteq P \subseteq P \subseteq G \). The polygon \( P \) may be made up of arbitrarily many (but finitely many) connected components and will be bounded by a finite number of simple closed polygonal curves \( \gamma \). For simplicity we shall deal in what follows with one of these, but with the understanding that the construction is carried out simultaneously for all of these polygonal curves.

We enclose each polygonal curve \( \gamma \) in a region \( K_\gamma \) homeomorphic to a plane ring and entirely lying in \( G - P_\varepsilon \). We take the regions \( K_\gamma \) for different \( \gamma \) to be nonoverlapping.

Suppose that \( 2\varepsilon_1 \) is the distance in the metric \( \rho \) from \( \gamma \) to the boundary of \( K_\gamma \), and that \( 2\varepsilon_2 \) is the lower bound for the lengths of curves in \( K_\gamma \) homologous to \( \gamma \).

We arrange on \( \gamma \) a finite system of points \( A_i \) in the following way. First we mark the vertices of \( \gamma \). Then on the sides of \( \gamma \) we mark off, two at a time, points distant from the endpoints by distances less than \( \min(\varepsilon_1, \varepsilon_2) \). Suppose further that \( 2\varepsilon_3 \) is the smallest of the distances from a link of the polygonal curve \( \gamma \), shortened from the endpoints, to the remaining links of \( \gamma \). We decompose the already shortened links into segments smaller than \( \min(\varepsilon_1, \varepsilon_2, \varepsilon_3) \). Finally, we enumerate all the points \( A \) thus marked off in cyclic order around the contour \( \gamma \).

We join the points \( A_i \) in each metric \( \rho_n \) successively by shortest arcs \( (A_iA_{i+1})_n \), drawing them so that two successive shortest arcs will have only one common endpoint or only a common segment adjacent to that point. Then we choose a subsequence \( n \) for which the thus-drawn shortest arcs \( (A_iA_{i+1})_n \) on each piece converge to some shortest arc \( A_iA_{i+1} \) in the metric \( \rho \). In what follows we shall suppose that the numeration of \( n \) refers only to this subsequence.

On a piece corresponding to one link of the polygonal curve \( \gamma \), the
shortest arcs $A_i A_{i+1}$, from Theorem 1 of this chapter, constitute a single shortest arc in the metric $\rho$, so that they \textit{a fortiori} form a simple curve. Now close to a vertex of the polygonal curve $\gamma$ two shortest arcs $A_i A_{i+1}$ merging at that vertex may have a common piece adjacent to the vertex, as depicted on Figure 90 near the points $B_1, B_2$. In Figure 90 the broken curves denote a polygonal curve $\gamma$ and the continuous lines the shortest arcs $A_i A_{i+1}$. Thus, the limiting shortest arcs $A_i A_{i+1}$ in the metric $\rho$ bound a polygon $P_0$ with the indicated possible singularities close to those points of $A_i$ which were vertices of $\gamma$.

The broken curves $(A_i A_{i+1})_n$ converge to the contour $P_0$ and for sufficiently large $n$ bound a polygon $P_n$ which may have singularities of the indicated kind at each of the vertices $A_i$, as in Figure 91.

6. We encircle the vertices of the polygon $P_0$ by nonintersecting neighborhoods $U_i$. We select and hold to some fixed subsequence of $n$ for which the quantity $\Omega_n(U_i)$ approaches a limit for each $i$. Since for each $n$
\[
\sum_i \Omega_n(U_i) \leq \Omega(G - F) < 2\varepsilon, \quad \sum_i \Omega(U_i) \leq \Omega(G - F) < 2\varepsilon,
\]
then we may suppose that $n$ is so large that
\[
\Omega_n(U_i) < \varepsilon_i, \quad \Omega(U_i) < \varepsilon_i,
\]
where $\sum \varepsilon_i \leq 4\varepsilon$.

Then, by Lemma 18 on close sectors with common vertices, for sufficiently large $n$ we will have
\[
(32) \quad |\tilde{\delta}(P_0) - \tilde{\delta}(P_n)| < \sum_i 9\varepsilon_i \leq 36\varepsilon.
\]

7. If the sides of the polygon issuing from the vertex $B$ coincide on an initial piece $BB'$, and thus form a "protuberant tail" or a "reentrant tail", then, dropping the piece $BB'$, we replace the angle $\tilde{\alpha}$ of the sector of the polygon at the vertex $B$ by a new sector angle $\tilde{\alpha}'$ at the vertex.
where in the case of a protuberant tail
\[ |\tilde{\alpha} - \tilde{\alpha}'| \leq \Omega(B') \]
and in the case of a reentrant tail
\[ |\tilde{\alpha} - \tilde{\alpha}'| \leq |\tilde{\alpha} - 2\pi| + |2\pi - \tilde{\alpha}'| \leq \Omega(B) + \Omega(B'). \]

Therefore, rejecting the tails from the polygons \(P_0, P_n\), we obtain polygons \(P_0', P_n'\) for which
\[
\begin{align*}
(33) & \quad |\tilde{\delta}(P_0') - \tilde{\delta}(P_0)| \leq \Omega_n(G - F) < 2\varepsilon, \\
(34) & \quad |\tilde{\delta}(P_n') - \tilde{\delta}(P_n)| \leq \Omega_n(G - F) < 2\varepsilon.
\end{align*}
\]

Moreover, from Theorem 9 of Chapter VI, the excesses computed with respect to the sector angles of a polygon differ from its curvature by no more than the curvature at its vertices and the rotations of its sides. Hence
\[
|\tilde{\delta}(P') - \omega_n(P')| \leq \Omega_n(Fr. P') \leq \Omega_n(G - F) < 2\varepsilon.
\]

We note further that
\[
|\omega_n(P'_n) - \omega_n(G)| = |\omega_n(G - P'_n)| \leq \Omega_n(G - F) < 2\varepsilon.
\]

Finally, by the last inequalities and inequalities (31)—(34) we have
\[
\begin{align*}
|\omega(G) - \omega_n(G)| & \leq |\omega(G) - \tilde{\delta}(P')| + |\tilde{\delta}(P') - \omega_n(P'|) + |\tilde{\delta}(P') - \tilde{\delta}(P_n)| \\
& \quad + |\tilde{\delta}(P_n) - \omega_n(P'|) + |\omega_n(P_n) - \omega_n(G)| \\
& < \varepsilon + 2\varepsilon + 36\varepsilon + 2\varepsilon + 2\varepsilon + 2\varepsilon + 2\varepsilon.
\end{align*}
\]

Thus
\[
|\omega(G) - \omega_n(G)| < 45\varepsilon,
\]
which with \(45\varepsilon < \varepsilon_0\) contradicts inequality (30). Thus Theorem 6 is proved.

**Remarks.** 1) If under the conditions of Theorem 6 the sequence \(\omega_n\) does not contain escaping loads, which *a fortiori* is the case for compact \(R\), then from Lemma 17 and Theorem 6 it follows that \(\omega_n \rightarrow \omega\) on all of \(R\).

2) Under the conditions of Theorem 6 the requirement of boundedness of the curvature of the metric \(\rho\) may be deduced from the uniform boundedness of the curvatures of the metrics \(\rho_n\) (see Chapter IV).

5. **Regular convergence.**

14. **Regularly converging metrics.** Earlier we have proved that under uniform convergence of metrics with absolute curvatures bounded uniformly, the curvatures \(\omega_n\) of these metrics converge locally weakly. But the positive and negative parts \(\omega^+_n, \omega^-_n\) of these curvatures may fail to
converge locally weakly, and therefore \textit{a fortiori} may fail to converge locally weakly to the functions $\omega^+, \omega^-$ for the limiting metric. We shall show this with a simple example.

Suppose that the surface $\Phi_n$ is an open plane square, at the middle $O$ of which there is a protuberance in the form of a lateral surface of a right circular cone with height $1/n$ and complete angle $\theta = \pi$ at the vertex. The metric $\rho_n$ is defined as the intrinsic of the surface $\Phi_n$. As $n \to \infty$ the surfaces $\Phi_n$ and their metrics $\rho_n$ converge uniformly to the plane square $\Phi$ with its metric $\rho$. Evidently, on the limit surface $\omega^+ = \omega^- = 0$. On the surfaces $\Phi_n$ the positive curvature is concentrated at the point $O$, where $\omega_n^+(0) = \omega_n(0) = 2\pi - \theta = \pi$, and the negative part of the curvature, $\omega_n^-$, is concentrated along the curve of the base of the conical protuberance, where it also adds up to $\pi$. In the limit $\omega_n^+$ and $\omega_n^-$ absorb each other so to speak, and, in spite of the convergence $\omega_n \to \omega$, there is no convergence $\omega_n^+ \to \omega^+$ or $\omega_n^- \to \omega^-.$

**Definition.** We shall say that the metrics $\rho_n$ with curvatures $\omega_n$, $\omega_n^+$, $\omega_n^-$, $\Omega_n$ converge regularly to the metric $\rho$ with curvatures $\omega$, $\omega^+$, $\omega^-$, $\Omega$, if $\rho_n$ converges locally uniformly to $\rho$ and if we have also

$$\omega_n^+ \to \omega^+, \quad \omega_n^- \to \omega^-.$$  

Evidently in this case also  

$$\omega_n \to \omega, \quad \Omega_n \to \Omega.$$  

In Theorem 7 we shall establish regular convergence under certain special approximations of metrics of bounded curvature. In this connection we shall find the following result of use.

**Lemma 19.** Suppose that the polygon $P$ in a metric $\rho$ of bounded curvature is subjected to a triangulation $Z$, i.e., is decomposed into triangles $t_i$. We replace each of the triangles $t_i$ by a plane triangle with sides of the same length. In accordance with the triangulation $Z$ there is constituted from the plane triangles a polyhedral development $P_Z$. Then

$$(35) \quad \omega_Z^+(P_Z) \leq \omega^+(P_\sim), \quad \omega_Z^-(P_Z) \leq \omega^-(P),$$

where $\omega^+(P_\sim)$ and $\omega_Z^+(P_\sim Z)$ are the positive parts of the curvature of the interior regions of $P$ and $P_Z$, and

$$\omega^-(P) = \omega^-(P_\sim) + \sum_i \tau_i^- (a_i),$$

$$\omega_Z^-(P_Z) = \omega_Z^-(P_\sim Z) + \sum_i \tau_i^-(a_{iZ}),$$
where $a_i$ are the sides of $P$, $a_{iz}$ polygonal curves in $P_z$ corresponding to the sides of $P$, $\tau_i$ the rotations from the side of $P$ and $P_z$.

**Proof.** At each interior vertex $A$ of the development $P_z$

\[(36) \quad \omega_z(A) = \omega(A) + \sum_j (\alpha_j - \alpha_j^0),\]

where $\bar{\alpha}_j$ are the angles of the sectors of the triangles $t_i$ adjacent to $A$, and $\alpha_j^0$ the corresponding angles in the plane triangles. We add the equations (36) over all those vertices $A^+$ at which $\omega_z(A) > 0$. Recalling from Theorem 11 of Chapter VI that

\[\sum_{A^+} \sum_j (\bar{\alpha}_j - \alpha_j^0) \leq \sum_j \omega^+(t_i - \gamma),\]

we thus obtain the first of inequalities (35).

For a vertex $B$ of the development $P_z$ which lies on the boundary of $P_z$ inside the polygonal curve corresponding to the side of $P$ we have

\[(37) \quad \tau_z(B) = \phi_z - \pi = \bar{\phi} - \pi + \sum_k (\bar{\alpha}_k - \alpha_k^0),\]

where $\phi_z$ and $\bar{\phi}$ are the sector angles of $P_z$ and $P$ at the vertex $B$, $\bar{\alpha}_k$ the angles of the sectors of the triangles $t_i$ adjacent to $B$, and $\alpha_k^0$ the corresponding angles of the plane triangles.

Add equations (36) over the interior vertices $A^-$ at which $\omega_z(A) < 0$, and equations (37) over the vertices $B^-$ at which $\tau_z(B) < 0$. Recalling from Theorem 11 of Chapter VI

\[\sum_{A^-} \sum_j (\bar{\alpha}_j - \alpha_j^0) + \sum_{B^-} \sum_k (\bar{\alpha}_k - \alpha_k^0) \geq - \sum_i \tilde{\omega}^-(t_i),\]

we obtain after a change of sign

\[- \sum_A \omega_z(A) - \sum_B \tau_z(B) \leq - \sum_A \omega(A) - \sum_B \tau(B) + \sum_i \tilde{\omega}^-(t_i),\]

from which follows the second of the inequalities (35).

Lemma 19 is proved.

**Theorem 7.** If the polyhedral metrics $\rho_n$, locally weakly converging to a metric $\rho$ of bounded curvature, are obtained from $\rho$ by rectifying triangles for ever finer triangulations $Z_n$, as was described in subsections 16, 17 of Chapter III and subsections 1–3 of Chapter IV, then $\rho_n$ converges regularly to $\rho$, i.e.,

\[\omega_n^+ \to \omega^+, \quad \omega_n^- \to \omega^-\]

We shall carry out all the discussion within the limits of an arbitrary
region $Q$ with compact closure. In making up the developments the region $Q$ is topologically mapped on a region in a space with a polyhedral metric. We shall suppose that the polyhedral metrics $\rho_n$ are simply given in the same region $Q$.

From Theorem 6, within the limits of the region $Q$ we have weak convergence $\omega_n \rightharpoonup \omega$. We assert that in the case at hand also $\omega_n^+ \rightharpoonup \omega^+$. From Lemma 11, for the proof of this fact it suffices to prove that for each open $G \subset \overline{G} \subset Q$ for which $\omega^+(\text{Fr. } G) = 0$ we have

\[
\lim_{n \to \infty} \omega_n^+(G) = \omega^+(G).
\]

But because of the general Lemma 9, from the weak convergence $\omega_n \rightharpoonup \omega$ it follows that

\[
\lim \inf_{n \to \infty} \omega_n^+(G) \geq \omega^+(G),
\]

so that it is sufficient for us to prove that if $\omega^+(\text{Fr. } G) = 0$ we have the reverse inequality

\[
\lim \sup_{n \to \infty} \omega_n^+(G) \leq \omega^+(G).
\]

Because of the complete additivity of $\omega^+$, for any $\varepsilon > 0$ in $Q$ there is a neighborhood $G_\varepsilon$ such that $\overline{G} \subset G_\varepsilon \subset Q$ and $\omega^+(G_\varepsilon - \overline{G}) < \varepsilon$. For sufficiently large $n$ the triangulations $Z_n$ are made so fine that from them we may make up polygons $P_n$ for which $\overline{G} \subset P_n(\subset \subset) \subset P_n \subset G_\varepsilon$. Then we will have

\[
\omega^+(G) = \omega^+(\overline{G}) > \omega^+(P_n(\subset \subset)) - \varepsilon.
\]

Moreover, as we know from Lemma 19, $\omega^+(P_n) \geq \omega_n^+(P_n(\subset \subset))$, so that the preceding inequality may be put in the form

\[
\omega^+(G) > \omega_n^+(P_n(\subset \subset)) - \varepsilon \geq \omega_n^+(G) - \varepsilon.
\]

From this last inequality, because of the arbitrary smallness of $\varepsilon > 0$, relation (40) follows, which along with inequality (39) gives equation (38) and proves Theorem 7.

**Theorem 8.** If under the conditions of Theorem 7 one and the same polygon $P$ undergoes the triangulations then

\[
\omega^+(P_\sim) = \lim_{n \to \infty} \omega_n^+(P_\sim),
\]

\[
\tilde{\omega}^-(P) = \lim_{n \to \infty} \tilde{\omega}_n^-(P),
\]

where

\[
\tilde{\omega}^-(P) = \omega^-(P_\sim) + \sum_i \tau^-(a_i) \quad (a_i \text{ a side of } P),
\]
\[ \bar{\omega}^- (P) = \omega_n^- (P-) + \sum_i \tau^- (a_{in}) \]

\((a_{in} - \text{polygons corresponding to the sides of } P)\).

**Proof.** From the general Lemma 9,

\[ \omega^+ (P-) \leq \lim \inf_{n \to \infty} \omega_n^+ (P_-), \]

and from Lemma 19 in the case at hand

\[ \omega^+ (P-) \geq \omega_n^+ (P_-). \]

(41) follows from these two inequalities.

From the generalized Gauss-Bonnet theorem,

(43) \[ \delta(P) = \omega(P_-) + \sum \tau(a_i) = \omega^+ (P-) - \omega^-(P-) - \sum_i \tau^- (a_i) = \omega^+ (P-) - \bar{\omega}^- (P^-). \]

Analogously, if in considering \( P \) in the metric \( \rho_n \) the \( a_{in} \) are regarded as complete "sides", then we will have

(44) \[ \delta_n(P) = \omega_n(P_-) + \sum_i \tau(a_{in}) = \omega^+ (P_-) - \omega^-(P_-) + \sum_i \tau^+ (a_{in}) - \sum_i \tau^- (a_{in}) \]

\[ = \omega^+ (P_-) - \bar{\omega}^- (P) + \sum_i \tau^+ (a_{in}). \]

But by the construction at each vertex of \( P \) we have convergence of the corresponding sector angles, so that

(45) \[ \lim_{n \to \infty} \delta_n = \bar{\delta}(P). \]

Moreover, the positive rotations \( \tau^+ (a_{in}) \) could appear only at the expense of a decrease in the sector angles for the swung triangles, adjacent to the boundary. These contractions do not exceed the positive curvature of the developed triangles, more precisely of their interior regions. But for large \( n \) we are dealing with triangles whose interior regions lie in vanishing neighborhoods of the boundary \( P \), where in view of the complete additivity of \( \omega^+ \) the values of \( \omega^+ \) become small. Therefore

(46) \[ \lim_{n \to \infty} \sum_i \tau^+ (a_{in}) = 0. \]

Finally, we have already proved equation (41).

From equations (43), (44), and the limiting relations (45), (46), and (41) follows the validity of relation (42). The theorem is proved.

**Remark.** Because of equation (46), we may consider \( \sum_i - \tau(a_{in}) \) in place of \( \sum_i \tau^- (a_{in}) \).

15. **Variation of the angle \( \gamma \).** The regular approximations described in subsection 14 are an important means of investigation of general metrics.
As an example we consider for any metric the variation of the angle \( \gamma \), which for the polyhedral metric was studied in § 2 of Chapter IV.

Suppose that \( T \) is a reduced triangle without interior tails in a two-dimensional manifold of bounded curvature. We shall take \( T \) to be convex, so that all distances are measured in \( T \) itself. This may always be done by pasting the triangle \( T \) into a plane in the place of the plane triangle \( T_0 \) having sides of the same length.

To each pair of points \( X \pm A, Y \pm A \) on the sides \( AB, AC \) of the triangle \( T \) there corresponds an angle \( \gamma(X, Y) \) at the vertex \( A_0 \) in the plane triangle with sides \( AX, AY, XY \).

**Theorem 9.** If the pairs of points \( X_i, Y_i \) (\( i = l, \ldots, r + 1 \)) form on the sides \( AB, AC \) of the triangle \( T \) an increasing sequence (i.e., \( X_{i+1} \in [X, B], Y_{i+1} \in [Y, C] \)), then the sum of the positive increments of the angle \( \gamma_T \) does not exceed \( \bar{\omega}^-(T) \):

\[
\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^+ \leq \bar{\omega}^-(T),
\]

and the sum of the absolute values of the negative increments does not exceed \( \omega^+(T) \):

\[
\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^- \leq \omega^+(T).
\]

**Proof.**

1. We carry out some triangulation \( Z_n \) of the triangle \( T \) into triangles of diameters less than \( d_n = 1/n \). Over these triangles we construct a polyhedral development \( Q_n \). From Lemma 19,

\[
\omega^+(Q_{\sim n}) \leq \omega^+(T_\sim), \quad \bar{\omega}^-(Q_n) \leq \bar{\omega}^-(T).
\]

2. The polygonal curve \( \widehat{AB} \), which corresponds in \( Q \) to the side \( AB \) of the triangle \( T \), might turn out not to be a shortest arc in \( Q_n \). However, from Theorem 9 of Chapter III, it cannot exceed in length that of a shortest arc in \( Q_n \) by more than \( \varepsilon < Cd_n \). In this case we construct a plane isosceles triangle with the sum of its lateral sides equal to \( \widehat{AB} \) and a base equal to a shortest arc in \( Q_n \). Suppose that \( \alpha_{AB} \) is the exterior angle at a vertex of this triangle. We paste this triangle along the lateral sides to the broken curve \( \widehat{AB} \) in \( Q_n \). We carry out if necessary an analogous pasting for the polygonal curves \( \widehat{AC} \) and \( \widehat{BC} \). We obtain a polyhedral triangle \( Q'_n \), in which the sides are already shortest arcs. The original broken curves \( AB, AC, BC \) compare uniformly with them in length. Here evidently we will have
(49) \[ \omega^+(Q'_n) \leq \omega^+(T^-) + 3\beta_n, \quad \omega^-(Q'_n) \leq \omega^-(T^-) \]

where

\[ \beta_n = \max(\alpha_{AB}, \alpha_{AC}, \alpha_{BC}). \]

3. If we carry out the indicated constructions for ever finer triangulations \( Z_n \), then the metrics in the developments \( Z_n \) will converge to the metric in \( T \). Therefore \( \gamma_0(X_{in}, Y_i) \rightarrow \gamma_T(X_i, Y_i) \).

4. Using for \( Q' \) Theorem 1 of Chapter IV, and also taking account of inequalities (49) and the fact that \( \beta_n \rightarrow 0 \), we obtain assertions (47) and (48) of Theorem 9.

Remark. It follows from Theorem 9 that for any monotone variation \( X(t), Y(t) \) from \( A \) towards \( B \) and from \( A \) towards \( C \) along \( AB, AC \) the function \( \gamma(t) = \gamma_T(X(t), Y(t)) \) has bounded variation, with

\[ \text{Var}^+\gamma(t) \leq \omega^-(T^-), \]
\[ \text{Var}^-\gamma(t) \leq \omega^+(T^-). \]

These results strengthen the assertions of Theorem 10 of Chapter VI. Analogously Theorem 9 may be used to extend Theorems 2 and 3 of Chapter IV to the case of nonpolyhedral metrics.


16. Angles of converging sectors. Suppose that within the limits of a two-dimensional region \( G \) there are defined metrics \( \rho_n, \rho \) of bounded curvature, with \( \rho_n \) uniformly converging to \( \rho \). Suppose moreover that in the metric \( \rho \) the points \( O_n \rightarrow O \) and the shortest arcs \( L_n, M_n \) issuing from the points \( O_n \) in the metrics \( \rho_n \) converge as curves in the metric \( \rho \) to shortest arcs \( L, M \) issuing from \( O \). We suppose moreover that each pair of shortest arcs \( L_n, M_n \) and \( L, M \) either has no common points besides a common endpoint \( O_n \) or an initial segment adjacent to it, and thus subdivide a small neighborhood of \( O_n, O \) into two sectors. Let \( V_n \) and \( V \) be distinguished sectors in the order of circuiting the vertex \( O_n \) or \( O \) from \( L_n \) or \( L \) to \( M_n \) or \( M \). Then \( V_n \rightarrow V \) in the sense of Definition 3' in subsection 3 of this chapter.

Let \( G^*_n \) be any neighborhood of the point \( O_n \) homeomorphic to the disc, with all the points of \( G^*_n \) distant from \( O_n \) by not more than \( \varepsilon \). Suppose that \( V^*_n \) is the portion of \( G^*_n \) related to the sector \( V_n \). We introduce the following characteristics of the subsequence \( \{V_n\} \):

\[ \omega^+(Q'_n) \leq \omega^+(T^-) + 3\beta_n, \quad \omega^-(Q'_n) \leq \omega^-(T^-) \]
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\[
\begin{align*}
\omega^+ \{ V_n \} &= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{V_n^*} \omega^+_n (V_n^*) , \\
\omega^- \{ V_n \} &= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{V_n^*} \omega^-_n (V_n^*) , \\
\bar{\omega}^- \{ V_n \} &= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{V_n^*} \bar{\omega}^-_n (V_n^*) ,
\end{align*}
\]

where

\[
\bar{\omega}^-_n (V_n^*) = \omega^-_n (V_n^*) + \tau_{in} (L_n \cap V_n^*) + \tau_{in} (M_n \cap V_n^*),
\]

\(\tau_{in}\) being the rotation on the side of \(V_n\) in the metric \(\rho_n\).

**Theorem 10.** If, as described above, \(\rho_n \to \rho\), \(O_n \to O\), \(L_n \to L\), \(M_n \to M\), \(V_n \to V\), then for the angles \(\bar{\alpha}_n\) and the sectors \(V_n, V\) the following relations hold:

\[
\lim_{n \to \infty} \sup \bar{\alpha}_n - \omega^+ \{ V_n \} \leq \bar{\alpha} \leq \lim_{n \to \infty} \inf \bar{\alpha}_n + \bar{\omega}^- \{ V_n \}. 
\]

**Proof.** First suppose that \(\bar{\alpha} < \pi\). Then the excision of the sector \(V\) from the space with metric \(\rho\) generates a metric \(\rho V\), in which the angle between the sides is equal to \(\bar{\alpha} = \min [\bar{\alpha}, \pi] = \bar{\alpha} < \pi\). Therefore for any points \(X, Y\) on the sides of \(V\) any shortest arc \(XY\) joining them in \(V\) cannot pass through \(O\). Let \(OX = x\), \(OY = y\), \(XY = z\), and \(\alpha^0(x, y)\) be the corresponding angle in the plane triangle with sides \(x, y, z\).

We excise analogously from spaces with metrics \(\rho_n\) sectors \(V_n\), and obtain in them induced metrics \(\rho_{V_n}\). From Theorem 4, the induced metrics converge uniformly: \(\rho_{V_n} \to \rho_V\). Suppose that \(X_n, Y_n\) are points on \(L_n, M_n\) corresponding in parameter on the shortest arcs to the points \(X, Y\) on \(L, M\). The relative shortest arcs \(X_nY_n\) in the sectors \(V_n\), beginning with some \(n\), cannot pass through \(O_n\), for otherwise the limit of a subsequence of these shortest arcs would lead to a shortest arc \(XY\) passing through \(O\).

We mark off \(X_nY_n\), excising from \(V_n\) reduced triangles \(T_n\), and we develop \(T_n\) onto the plane. Then from Theorem 10 of Chapter VI,

\[
\bar{\alpha}_n - \omega^+ (T_{(\epsilon^{-}\gamma)n}) \leq \alpha^0_n (x, y) \leq \bar{\alpha}_n + \bar{\omega}^- (T_n).
\]

For \(x\) and \(y\) small in comparison with \(\epsilon\) we have moreover

\[
\bar{\alpha}_n - \omega^+ (V_{(\epsilon^{-}\gamma)n}) \leq \alpha^0_n (x, y) \leq \bar{\alpha}_n + \bar{\omega}^- (V_n^*).
\]

We pass in the left inequality to the limit through the same subsequence of \(n\) for which \(\lim \sup \bar{\alpha}_n\) is realized, and in the right inequality through that subsequence for which \(\lim \inf \bar{\alpha}_n\) is realized. Then we strengthen these inequalities by passing to the least upper bounds with respect to
and the lower limit as $n \to \infty$ in the remaining terms. Finally, we let $x, y,$ and $\varepsilon$ tend to zero. Then, recalling that $\lim_{x,y\to0} \alpha^0(x,y) = \bar{\alpha} = \bar{\alpha}$, we obtain inequality (52).

Now we suppose that $\bar{\alpha} > \pi$. We subdivide the sector $V$ by a finite number of shortest arcs $L_k$ into sectors with angles $\bar{\alpha}^k$, with $0 < \bar{\alpha}^k < \pi$. Then we choose a $\beta$ for which the condition $0 < 2\beta < \bar{\alpha}^k < \pi - 2\beta$ is satisfied for all $k$. Finally, on each of the shortest arcs $L_k$ we mark a point $A^k$ so close to $O$ that no pair of shortest arcs $\overline{AO}, \overline{A'O}$ can form at the vertex $O$ an angle larger than $\beta$. For sufficiently large $n$ all the points $A^k$ fall inside $V^*_n$. In what follows we will suppose that this construction is carried out separately for the subsequences of $n$ realizing $\limsup n \to \infty \bar{\alpha}_n$ and $\liminf \bar{\alpha}_n$. We choose a subsequence for which all the $L_k$ converge to some shortest arc $\overline{AO} = L^*$. These shortest arcs may fail to coincide with segments of the shortest arcs $L_k$, but because of the special choice of the points $A^k$ they also will subdivide the sector $V$ into sectors with the angles $0 < \bar{\alpha}^k < \pi$, while these sectors will be limiting for the sectors $V^*_n$.

Because the characteristics $w^+ \{V^*_n\}, \bar{\omega}^- \{V^*_n\},$ in distinction from $w^+ \{V^*_{n'}\}, \bar{\omega}^- \{V^*_{n'}\}$, are not additive under combination of sectors, we may not use directly the result (52) for separate sectors $\bar{\alpha}^k$. But we turn to inequalities (53), true for each $k$:

$$\bar{\alpha}_n - \omega^+_n(T^*_{n'}{\bar{\alpha}}_n) \leq \alpha^0_n(x, y) \leq \bar{\alpha}_n + \bar{\omega}^-(T^*_{n'}) .$$

For simplicity we shall suppose that for each $k$ we have chosen one and the same $x = y$. Adding these inequalities with respect to $k$ and using the additivity of $\omega^+$ and $\bar{\omega}^-$ under adjunction of the $T^*_{n}$ along entire sides, we obtain

$$\bar{\alpha}_n - \omega^+_n(V^*_{n'}{\bar{\alpha}}_n) \leq \sum_k \alpha^0_n(x, y) \leq \bar{\alpha}_n + \bar{\omega}^-(V^*_{n'}).$$

Passing as in inequality (54) to the limit as $n \to \infty$ on the left through the subsequence realizing $\limsup \bar{\alpha}_n$ and on the right $\liminf \bar{\alpha}_n$, then letting $x, y$ and $\varepsilon$ tend to zero, and taking account of the fact that $\sum_k \bar{\alpha}^k = \sum_k \bar{\alpha}^k = \bar{\alpha}$, we obtain also in this case inequality (52).

Theorem 10 is proved.

**Corollary 1.** If we denote by $W_n$ and $W$ the sectors which serve as complete neighborhoods of the points $O_n, O$ with the corresponding points $O_n, O$ deleted, then evidently

$$(55) \limsup_{n \to \infty} \bar{\alpha}_n - \omega^+ \{W_n\} \leq \bar{\alpha} \leq \liminf_{n \to \infty} \bar{\alpha}_n + \omega^- \{W\} .$$
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Corollary 2. For the complete angles \( \theta_n, \theta \) around the points \( O_n, O \) the following inequality is also valid:

\[
\limsup_{n \to \infty} \theta_n - \omega^+ \{ W_n \} \leq \theta \leq \liminf_{n \to \infty} \theta_n + \omega^- \{ W_n \}.
\]

Theorem 11. Under the hypotheses of Theorem 10, i.e., as \( \rho_n \to \rho, O_n \to O, L_n \to L, M_n \to M \), where \( L_n, M_n \) and \( L, M \) bound sectors with vertices \( O_n, O \), for the angles \( \alpha_n, \alpha \) between \( L_n, M_n \) and \( L, M \) the following inequality holds:

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ \{ W_n \} \leq \alpha \leq \liminf_{n \to \infty} \alpha_n + \omega^- \{ W_n \}.
\]

Proof. Suppose that \( \bar{\alpha}^1, \bar{\alpha}^2 \) and \( \bar{\alpha}^1, \bar{\alpha}^2 \) are the angles of the corresponding sectors with the sides \( L_n, M_n \) and \( L, M \). We know that

\[
\alpha_n = \min \{ \bar{\alpha}^1, \bar{\alpha}^2, \pi \}, \quad \alpha = \min \{ \bar{\alpha}^1, \bar{\alpha}^2, \pi \},
\]

so that it is not difficult to obtain Theorem 11 from Theorem 10.

First we prove the right inequality (57). If \( \liminf_{n \to \infty} \alpha_n = \pi \), then that inequality is trivial. Suppose \( \liminf_{n \to \infty} \alpha_n < \pi \). Then in the sequence realizing \( \liminf \alpha_n \), beginning with some \( n \), all \( \alpha_n < \pi - \epsilon \). In this sequence we may select a subsequence \( n_k \) for which some of the sectors, which we will denote by \( \bar{\alpha}_n \), satisfy \( \bar{\alpha}_n = \alpha_n \). Then

\[
\alpha \leq \bar{\alpha}^1 \leq \liminf_{k \to \infty} \bar{\alpha}_n + \omega^- \{ W_n \} = \liminf_{n \to \infty} \alpha_n + \omega^- \{ W_n \},
\]

which proves the right inequality in (57).

The proof of the left inequality (57) is somewhat more complicated. We hold to that sequence of \( n \) for which \( \limsup_{n \to \infty} \alpha_n \) is realized.

If that limit is equal to \( \pi \), then for any \( \epsilon > 0 \) beginning with certain \( n \), all \( \alpha_n \geq \pi - \epsilon \), \( \bar{\alpha}^1 \geq \pi - \epsilon \), \( \bar{\alpha}^2 \geq \pi - \epsilon \). Then from Theorem 10

\[
\pi - \epsilon - \omega^+ \{ W_n \} \leq \bar{\alpha}^1,
\]

\[
\pi - \epsilon - \omega^+ \{ W_n \} \leq \bar{\alpha}^2,
\]

\[
\pi - \epsilon - \omega^+ \{ W_n \} \leq \pi,
\]

from which, from (58), we have

\[
\pi - \epsilon - \omega^+ \{ W_n \} \leq \alpha,
\]

which, because of the arbitrariness of \( \epsilon > 0 \), yields the left inequality (57).

If now \( \limsup_{n \to \infty} \alpha_n < \pi \), then, beginning with some \( n \), all the \( \alpha_n < \pi \). In the sequence realizing \( \limsup \alpha_n \) we select a subsequence of the \( n_k \) for which \( \bar{\alpha}_n = \alpha_n \). Then from Theorem 10

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ \{ W_n \} = \limsup_{n \to \infty} \bar{\alpha}_n + \omega^+ \{ W_n \} \leq \bar{\alpha}^1,
\]

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ \{ W_n \} = \limsup_{k \to \infty} \bar{\alpha}_n - \omega^+ \{ W_n \} \leq \limsup_{k \to \infty} \bar{\alpha}_n - \omega^+ \{ W_n \} \leq \bar{\alpha}^2,
\]
so that the left inequality in (57) follows by (58).

Theorem 11 is proved.

17. Special cases of convergence of angles. The assertions enumerated below follow directly from Theorems 10 and 11 and the complete additivity of $\omega^+$ and $\omega^-$.

**Corollary 1.** If $\omega^+\{W_n\} = \omega^-\{W_n\} = 0$, then from inequalities (55)—(57) it follows that in this case

\[
\bar{\alpha}^1 = \lim_{n \to \infty} \bar{\alpha}_n^1, \quad \bar{\alpha}^2 = \lim_{n \to \infty} \bar{\alpha}_n^2, \quad \theta = \lim_{n \to \infty} \theta_n, \quad \alpha = \lim_{n \to \infty} \alpha_n.
\]

**Corollary 2.** If all the shortest arcs $L_n, M_n$ are drawn in one and the same metric, i.e., $\rho_n \equiv \rho$, and moreover from one and the same point, then from the complete additivity of $\omega^+$ and $\omega^-$ it follows that the hypotheses of Corollary 1 are satisfied, and therefore (59).

**Corollary 3.** If $\rho_n \equiv \rho$ and $\omega(0) = 0$, then the hypotheses of Corollary 1 are also satisfied and (59) is true.

**Corollary 4.** If $\rho_n \equiv \rho$ and all the sectors $V_n$ do not contain the point $O$ in their interior or on their sides, then because of the complete additivity of the curvature $\omega^+\{V_{(-\infty)}\} = \omega^-\{V_n\} = 0$ and therefore (52) implies

\[
\bar{\alpha} = \lim_{n \to \infty} \bar{\alpha}_n.
\]

**Corollary 5.** If $\rho_n \equiv \rho$ and all the sectors $V_n$ contain the point $O$ inside themselves, then the hypotheses of Corollary 4 are satisfied for the complementary sectors. For these inequality (52) is satisfied. Hence for the angles of the sectors $V_n, V$ themselves, we have

\[
\bar{\alpha} = \lim_{n \to \infty} \bar{\alpha}_n - \omega(O).
\]

**Corollary 6.** If $\rho_n \equiv \rho$ and $O$ might lie on the boundary of $V_n$, then always $\omega^+\{V_n\} = 0$, $\bar{\omega}^-\{V_n\} \leq -\omega(O)$, so that if $\rho_n \equiv \rho$ always

\[
\lim_{n \to \infty} \sup \bar{\alpha}_n \leq \bar{\alpha} \leq \lim_{n \to \infty} \inf \bar{\alpha}_n - \omega(O).
\]

**Corollary 7.** If all the metrics $\rho_n$ have only positive curvature, then $\bar{\omega}^-\{V_n\} = 0$ and it follows from (52) that

\[
\bar{\alpha} \leq \lim_{n \to \infty} \inf \bar{\alpha}_n.
\]

If all the $\rho_n$ are of negative curvature, then $\omega^+\{V_n\} = 0$ and

\[
\bar{\alpha} \geq \lim_{n \to \infty} \sup \bar{\alpha}_n.
\]
1. Triangles in polyhedral metrics.

1. Preparatory remarks. Suppose that $T$ is a triangle in a polyhedral metric. Excising $T$ and pasting it into the plane in the place of the plane triangle having sides of the same length, we may consider $T$ as a convex triangle in the resulting new polyhedral metric.

Lemma 1. If the convex triangle $T$ in a polyhedral metric contains within itself no vertex of positive curvature, then the diameter $d$ of the triangle $T$ is equal to the length of the largest of its sides.

Proof. In the absence of vertices of positive curvature any two points may be joined in $T$ by a unique shortest arc, and each shortest arc may be prolonged to the boundary in $T$. Otherwise there would be a two-gon in $T$, in which there must be at least one vertex of positive curvature. Suppose $X, Y \in T$ are those points for which $\rho(X, Y) = d$. From what has been said above it follows that $X$ and $Y$ lie on the boundary of $T$. Suppose that one of these points, say $T$, does not coincide with a vertex of $T$. Then at the point $Y$ the shortest arc $XY$ forms with the side $AC$ angles $\xi + \eta \geq \pi$, as depicted in Figure 92. Suppose for definiteness that $\eta \geq \pi/2$: Shifting $Y$ along the boundary of $T$ a short distance to a close position $Y'$, we increase the length of $XY$, thus contradicting the equation $\rho(X, Y) = d$. Consequently $X$ and $Y$ coincide with vertices of $T$, and Lemma 1 is proved.

Lemma 2. Suppose that a two-gon $D$ is excised from a compact region $\bar{G}$ in a two-dimensional manifold of bounded curvature. On identifying the edges of $D$ we obtain a new region $\bar{G}'$. When this is done the diameter of the region does not increase.
\[ d(G') \leq d(G) \]

**Proof.** In studying the diameters it suffices to consider the shortest possible curves joining pairs of points. The length of each curve in \( G \) which does not pass within \( D \) is preserved in \( G' \). Suppose that a curve joining the points \( X \) and \( Y \) has a piece \( MN \) in \( D \), as in Figure 93. We then may suppose that \( MN \) is a shortest arc in \( D \). Then from the triangle \( MON \) follows \( MN + NO \geq MO \), so that \( MN' = MO - NO \leq MN \), so that after excision of \( D \) the points \( X, Y \) are joined by a curve \( XM(N'M)Y \) in \( G' \) of no greater length than \( XMNY \). Hence \( d(G') \leq d(G) \).

**Lemma 3.** Suppose that a “triangle” \( T \) is situated on the plane, its sides being broken curves convex toward the interior of \( T \). (Not excluding the possibility that the pieces of the sides adjacent to a vertex may coincide, as depicted on Figure 92 at the vertex \( B \).) Suppose that \( \sigma(T) \) is the area of \( T \) and \( \sigma(T_0) \) the area of the ordinary plane triangle with sides of the same lengths as the broken curves serving as the “sides” of \( T \). Then

\[ 0 \leq \sigma(T_0) - \sigma(T) \leq \frac{1}{2} \omega^- d^2, \]

where \( \omega^- \) is the total negative rotation of the sides of \( T \), in absolute value, and \( d \) is the intrinsic diameter of \( T \).

**Proof.**
1. Excision of \( T \) induces in \( T \) an intrinsic metric in which the distance between vertices coincides with the lengths of the “sides” of \( T \). Therefore these three lengths satisfy the triangle inequality and \( T_0 \) may be constructed.
2. As shown in sub-
section 9 of Chapter II, in a plane quadrilateral with reentrant angle \( \delta \) and three salient angles \( \alpha, \beta, \gamma \) as in Figure 94, if we rectify the reentrant angle \( \delta \) each of the angles \( \alpha, \beta, \gamma \) increases. Moreover, for a small partial increase \( \Delta \delta \) we will have all of \( \Delta \alpha, \Delta \beta, \Delta \gamma > 0 \). This is easily verified by first straightening the polygon \( CDM \) (Figure 94) and then forming the broken curve \( BMA \).

Even a slightly rectified quadrilateral contains the original quadrilateral, cut along \( AD \), as shown in Figure 95. Therefore the increase of the area \( \Delta \sigma > 0 \). With the notations indicated in Figure 95, we have for a differential change of area

\[
d\sigma = \frac{1}{2} a^2 d\alpha + \frac{1}{2} b^2 d\beta + \frac{1}{2} c^2 d\gamma.
\]

Hence

\[
(2) \quad d\sigma \leq \frac{1}{2} (d\alpha + d\beta + d\gamma) [\max(a, b, c)]^2 = \frac{1}{2} \, d\delta [\max(a, b, c)]^2.
\]

3. Now we shall prove Lemma 3. Rectification of \( T \) into \( T_0 \) may be carried out by rectifying in turn the reentrant angles at the vertices of the polygons serving as the "sides" of \( T \). From Lemma 1, the diameter of the transformed triangle remains equal to the length of the largest of its sides. Therefore all the time

\[
d\sigma \leq \frac{1}{2} d^2 d\delta.
\]

But \( d\delta > 0 \) and \( d\delta = -d\omega^- \). Therefore inequality (1) follows from (2).

Lemma 3 is proved.

3. **Comparison with plane triangles.** Suppose that \( T \) is a triangle in the polyhedral metric, \( \omega^+ \) the total positive curvature of the vertices lying within \( T \), \( \omega^- \) the absolute value of the total negative curvature of the vertices lying within \( T \), augmented by the rotation of the sides of \( T \) on the side of the triangle. The area \( \sigma(T) \) is defined as the total area of all the plane pieces of which \( T \) consists, and \( \sigma(T_0) \) the area of the plane triangle \( T_0 \) with sides of the same length as \( T \).

**Theorem 1.** For every triangle \( T \) in the polyhedral metric

\[
(3) \quad \frac{1}{2} \omega^- d^2 \leq \sigma(T) - \sigma(T_0) \leq \frac{1}{2} \omega^+ d^2,
\]
where \( d \) is the intrinsic diameter of \( T \).

**Proof.** Without loss of generality we may suppose \( T \) convex. We consider separately four cases.

**Case 1.** If there is no vertex of the metric inside \( T \), Theorem 1 follows from Lemma 3. In this case \( T \) is isometric to the figure indicated in Lemma 3.

**Case 2.** Suppose that within \( T \) there are only vertices with negative curvature. From the connection between the curvature and the rotation of the sides with the excess of the triangle,

\[
\delta(T) + \omega^+(T_-) - \omega^-(T_-) - \sum_{i=1}^{3} \tau_i^- = \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} - \pi \geq -\pi
\]

we conclude that if \( \omega^+(T_-) = 0 \) always \( \omega^-(T_-) \leq \pi \). Therefore for any of the \( n \) interior vertices \( O \) we have \( -\pi \leq \omega(O) < 0 \).

If \( n = 0 \) the theorem is already proved. Suppose that it is true for \((n-1)\) vertices. We shall prove it for \( n \) vertices. Suppose that \( O \) is the closest of these vertices to \( A \). Draw the shortest arc \( AO \) and extend it through the point \( O \) so that it bisects the angle \( 2\pi + 2\alpha \) of the complete sector around the point \( O \) and, further, in passing any vertices with negative curvature leaves a sector \( \pi \) to the right of itself. Thus we prolong \( AO \) as a shortest arc \( AP \), where \( P \) lies on the boundary of \( T \). (We observe that \( AP \) remains a shortest arc and \( P \neq B, P \neq C \), for otherwise there would be a two-gon containing inside itself vertices with curvature \( \omega^+ > 0 \).) We make a cut along \( AP \) and paste along it the plane quadrilateral depicted in Figure 96 with sides of lengths \( O_1A = O_2A = OA, O_1P = O_2P = OP \) and obtuse angles \( \pi - \alpha \) at the vertices \( O_1, O_2 \).

As is clear from Figure 96, the pasted area

\[
\Delta \sigma < \frac{1}{2} AP^2 \xi < \frac{1}{2} (AO + OP)^2 \xi \leq \frac{1}{2} \xi d^2 = \frac{1}{2} \Delta \delta^+ d^2.
\]

where \( d \) is the diameter of \( T \), which from Lemma 1 does not change under the pasting. After the pasting the number of vertices decreases by exactly one, which along with the induction hypothesis and the inequality \( \Delta \sigma < (1/2) \Delta \delta^+ d^2 \) proves Theorem 1 for Case 2.

**Case 3.** Suppose that inside \( T \) there are also vertices of the metric with \( \omega^+ > 0 \). Suppose that some pair of vertices of the triangle \( T \) may
be joined in $T$ by a shortest arc distinct from a side of $T$. Then there exists a two-gon $D_1$ with vertices $M,N$, (Figure 97) which does not contain other shortest arcs $MN$. Inside $D_1$ there is at least one vertex of the metric with $\omega^+ > 0$.

The proof of the theorem in Case 3 will be carried out by induction on the number $n$ of vertices with $\omega^+ > 0$ inside $T$. Suppose that $O_1$ is one such vertex, lying in $D_1$. If $O_1$ can be joined in $D_1$ to $M$ by more than one shortest arc, then in $D_1$ there will appear a new two-gon $D_2$ with vertices $M$ and $O_1$. Again inside this there is a vertex $O_2$ with $\omega^+ > 0$. Thus in a finite number of steps we arrive at a vertex $O_k$ with $\omega^+ > 0$ which is joined to $M$ in the corresponding two-gon $D_k \subset D$ by a unique shortest arc $MO_k$.

Continue the segment $O_kS$ so that it bisects along with $MO_k$ the complete angle of the sector around $O$, as in Figure 98. For small $O_kS$ the point $S$ will be joined in $D_k$ to $M$ by exactly two shortest arcs, which pass to the right and left of $MO_k$. Suppose that $R'$ and $R''$ are the points at which these shortest arcs depart from $MO_k$. If we excise the two-gon $MR'SR''M$ thus formed and identify its edges, then the vertex $O_k$ is moved into the point $S$, the positive curvature decreases by no less than $\alpha + \beta$ (Figure 98), and the area of the polygon decreases by the sum of the areas of the two plane triangles $SR'O_k, SR''O_k$. Since the sides of these triangles do not exceed the diameter of the whole figure, and the latter from Lemma 2 remains for such excisions less than or equal to $d$, where $d$ is the intrinsic diameter of the original triangle, therefore for each such excision the inequality

$$-\Delta \sigma \leq \frac{1}{2} (-\Delta \omega^+)d^2$$

will be preserved.

Repeating such excisions a finite number of times, we will sooner or later liquidate the entire two-gon $D$. This process will certainly decrease the number of vertices with $\omega^+ > 0$. Taking account of the induction hypothesis and inequality (4) we establish Theorem 1 for this case.
Case 4. We suppose finally that inside $T$ there are vertices of the metric with $\omega^+ > 0$ and vertices of $T$ not joinable in $T$ by shortest arcs distinct from the sides of $T$. Then we consider in $T = ABC$ the closest vertex $O_1$ to $A$ with $\omega^+ > 0$. As in Case 3, we carry out excisions of the corresponding two-gons. This time it may happen that further excisions of the two-gon are not possible, because there appears in $T$ a shortest arc distinct from the sides of $T$. But then we fall into the conditions of Case 3.

Inequality (4) is observed under each excision. This completes the proof of Theorem 1.

2. Definition of area.

3. Area of a polygon.

Definition. The area $\sigma_0(P)$ of a polygon $P$ is the limit of the sums $\sum_{t_i \in Z} s(t'_i)$ of areas of plane triangles with sides of the same length as the triangles $t_i$ of triangulations $Z$ of the polygon $P$, the limit being taken under the condition that the triangulations $Z$ become ever finer:

$$\sigma_0(P) = \lim_{\max_a(d(t_i)) \to 0} \sum_{t_i \in Z} s(t'_i).$$

Lemma 4. Suppose that $T$ is a triangle, $\omega^+$ and $\omega^-$ the positive and negative parts of its curvature, the latter being augmented by the rotations of its sides turned towards $K$, and $d$ its intrinsic diameter. Then for any $\varepsilon > 0$ there exists an arbitrarily fine triangulation of the triangle $T$ under which the total area $s(Q)$ of the plane triangles in the polyhedral development $Q$ corresponding to $Z$ satisfies the inequality

$$-\frac{1}{2} \omega^- d^2 + \varepsilon < s(Q) - s(T^0) < \frac{1}{2} \omega^+ d^2 + \varepsilon,$$

where $T^0$ is a plane triangle with sides of the same length as the triangle $T$.

Proof. Suppose that $x, y,$ and $z$ are the lengths of the sides of $T$. Taking an $\varepsilon > 0$, we choose a $\delta > 0$ such that the area of the plane triangle $T^0$ with sides $x, y, z$ differs from the area of the plane triangle $T^0$ with sides $x \cos \delta, y \cos \delta, z \cos \delta$ by less than $\varepsilon/3$, and moreover the total area $s(t_1 + t_2 + t_3)$ of the three isosceles triangles $t_1, t_2, t_3$ with angles $\delta$ at the bases and lateral sides equal respectively to $x/2, y/2, z/2$ satisfies the condition

$$s(t_1 + t_2 + t_3) < \frac{\varepsilon}{3}.$$

For a fixed $\delta > 0$ we may in accordance with Lemma 10 of Chapter VI
construct an arbitrarily fine triangulation of the triangle $T$ such that the corresponding polyhedral development $Q$ after pasting the triangles $t_1, t_2, t_3$ to it will form a polyhedral development $T$ in which the bases of $t_1, t_2,$ and $t_3$ will be shortest arcs.

We may take it for granted that

$$\omega^+(R) \leq \omega^+(Q) + 6\delta \leq \omega^+(T) + 6\delta,$$

$$\bar{\omega}^-(R) \leq \bar{\omega}^-(Q) \leq \bar{\omega}^-(T).$$

Here in each row the first inequality follows trivially from the structure of the pasting and the second from Theorem 11 of Chapter VI. Moreover, on refinement of the development $Q$ we see that the diameter $d(Q) \to d(T)$ and for sufficiently small $\delta$ this diameter differs little from $d(R)$. Therefore we may suppose that $\delta$ is so small and the triangulation with respect to which $Q$ was constructed so fine that

$$\frac{1}{2} \omega^+(R)d^2(R) \leq \frac{1}{2} \omega^+(T)d^2(T) + \frac{\varepsilon}{3},$$

$$\frac{1}{2} \bar{\omega}^-(R)d^2(R) \leq \frac{1}{2} \bar{\omega}^-(T)d^2(T) + \frac{\varepsilon}{3}.$$

Now for the proof of Lemma 4 it is sufficient to apply Theorem 1 to the development $R$:

$$(7) \quad -\frac{1}{2} \bar{\omega}^-(R)d^2(R) \leq s(R) - s(T^o) \leq \frac{1}{2} \omega^+(R)d^2(R).$$

Replacing in (7) $s(R)$ by $s(Q)$, $s(T^o)$ by $s(T^o)$, $1/2\omega(R)d^2(R)$ by $1/2\omega(T)d^2(T)$, we each time make an error less than $\varepsilon/3$. Therefore (6) follows from inequality (7). Lemma 4 is proved.

We note that the construction of the triangulation may without loss of property (5) be constructed so that it is a finer subdivision of any given triangulation.

**Theorem 2.** Every polygon $P$ has a definite area $\sigma_0(P)$, i.e., the limit (5) exists.

**Proof.** Suppose that $P$ is a polygon and $Q_1, Q_2$ polyhedral developments constructed with respect to triangulations $Z_1, Z_2$ of $P$. If in these triangulations the sides have multiple intersections, then this situation may be avoided without changing the structure of the triangulation nor the lengths of the sides of the triangles (i.e., without changing $Q_1, Q_2$) and if the greatest of the intrinsic diameters of the triangles of the triangulations $Z_1, Z_2$ is increased then this happens only a finite number of times. After this we
may carry out a common subdivision of the resulting $Z'_1, Z'_2$ and a finer triangulation $Z_3$ satisfying for each triangle $T$ of $Z'_1, Z'_2$ the requirements of Lemma 4. Then we will have

$$|s(Q_1) - s(Q_3)| \leq \frac{1}{2} \Omega(P)d^2(Z'_1) + \varepsilon,$$

$$|s(Q_2) - s(Q_3)| \leq \frac{1}{2} \Omega(P)d^2(Z'_1) + \varepsilon,$$

where $d(Z'_1) = \max_{T \in Z'_1} d(T)$ is the "diameter" of the triangulation $Z'_1$, so that, in view of the arbitrary smallness of $\varepsilon > 0$, the boundedness of $\Omega(P)$, and the fact that $d(Z'_1)$ tends to zero along with $d(Z_i)$, it follows that the limit (5) exists.

Theorem 2 is proved.

**Theorem 3.** If $P$ is a polygon and $Q_z$ the development obtained by rectification of the triangles of the triangulation $Z$ of the polygon $P$ with intrinsic diameters no greater than $d(Z)$, then

(8) $$\frac{1}{2}\omega^-(P)d^2(Z) \leq \sigma_0(P) - s(Q_z) \leq \frac{1}{2}\omega^+(P)d^2(Z).$$

Here it may happen that the structure of $\omega^+(P)$ does not include the curvatures of points which serve as vertices of the triangulation, but that the curvatures of such points and the rotations of the sides of $P$ at such points are included in $\tilde{\omega}^-(P)$.

The assertion of Theorem 3 follows immediately from Lemma 4 and Theorem 2.

4. Area of arbitrary sets.

**Definition.** By the area $\sigma(M)$ of an arbitrary set $M$ we shall mean the quantity

(9) $$\sigma(M) = \inf_{G \supseteq M} \sup_{P \subseteq G} \sigma_0(P),$$

where $G$ are open sets and $P$ polygons. Disconnected polygons are admitted. From definition (9) it follows that $\sigma(M)$ is a negative monotone (with respect to set inclusion) function. The values $\sigma = \infty$ are not excluded. For open sets we evidently have

(10) $$\sigma(G) = \sup_{P \subseteq G} \sigma_0(P).$$

It follows from Theorem 2 that all the conditions of Theorem 2 of Chapter V are satisfied for $\sigma(M)$, so that the following important theorem is valid.
DEFINITION OF AREA

Theorem 4. The area $\sigma(M)$ defined by equation (9) is a regular Carathéodory measure. In particular, it is completely additive on the ring of Borel sets.

Lemma 5. For every set $M$ which lies strictly inside a polygon $P$, i.e., $M \subset P$, the inequality

$$\sigma(M) \leq \sigma_0(P)$$

holds.

Indeed, suppose that the polygon $P_1 \subset P$. Then any triangulation of $P_1$ may be complemented to a triangulation of $P$, so that $\sigma_0(P_1) \leq \sigma_0(P)$. It follows from this that $\sigma(M) \leq \sigma(P_1) = \sup_{P_1 \subset P} \sigma_0(P_1) \leq \sigma_0(P)$, i.e., inequality (11).

Theorem 5. The areas $\sigma$ of one-point sets, of shortest arcs, of polygonal curves, and of geodesics are all equal to zero.

Proof. 1. Suppose that $A$ is a one-point set, and $\theta$ the complete angle around $A$. From Theorem 1 of Chapter III, $A$ may be enclosed in a convex polygon $P$ homeomorphic to the disc and lying in a neighborhood $U$ of radius $r$ of the point $A$. The shortest arcs going from $A$ to the vertices of $P$ decompose $P$ into triangles. On swinging these onto the plane the total area $s(Q)$ of the resulting development $Q$ will satisfy the inequality

$$s(Q) \leq \frac{1}{2} [\theta + \omega^+(U)] r^2,$$

and moreover, from Theorem 3,

$$|s(Q) - \sigma_0(P)| \leq \frac{1}{2} \Omega(U)(2r)^2.$$

Since $r$ may be taken arbitrarily small without decreasing $\theta$, $\omega^+(U)$, $\Omega(U)$, we therefore conclude that $A$ can be imbedded in a polygon with arbitrarily small area $\sigma_0(A)$. Therefore it follows from Lemma 5 that $\sigma(A) = 0$.

2. Suppose given an arbitrary shortest arc. It may be decomposed into a finite number of pieces on each of which its curvature and right and left rotations (all three quantities are not larger than zero) do not exceed a certain $\varepsilon_1 > 0$ in absolute value. In view of the additivity of $\sigma$ it is sufficient to prove that on each such piece $L$ the area $\sigma(L) = 0$.

Fix a small $\varepsilon_2 > 0$ and encircle $L$ by a region $U$ homeomorphic to the disc within which $\Omega(U - L) < \varepsilon_2$. Then divide $L$ by points $A'$ into segments
of small length \( r \).

If \( \varepsilon_1 \) is sufficiently small, then around each point \( A^i \) the complete angle is close to \( 2\pi \) and from it on each side of \( L \) we can pass two shortest arcs \( l^i_1, l^i_2 \) (Figure 99) such that they form with \( l \) and with one another sectors with angles close to \( \pi/6 \). Moreover, if \( r \) is small, we may suppose \( U \) so small that all these shortest arcs leave the limits of \( U \). Moreover, because of the sufficient smallness of the original \( \varepsilon_1 \) and the rational compatibility of the choice of \( \varepsilon_2, r \) and the width of \( U \) we can arrange things so that the shortest arcs \( \ldots, l^i_1, l^{i+1}_1, \ldots \) of one type do not intersect the shortest arcs \( \ldots, l^i_2, l^{i+1}_2, \ldots \) within the limits of \( U \) and each shortest arc \( l^i_1 \) with \( i > 2 \) intersects at least two of the preceding shortest arcs \( l^{i-1}_2, l^{i-2}_2 \).

In the resulting quadrilaterals \( A^{i-1}B^i C^iB^{i-1}A^{i-1} \) (Figure 99) we draw diagonals \( B^{i-1}B^i \). For sufficiently small \( \varepsilon_1 \), we may for arbitrarily small \( \varepsilon_2, r \) and width of \( U \) attain in fact the situation depicted in Figure 99, with the triangles \( A^{i-1}A^iB^i, B^{i-1}A^{i-1}B^i \) close to equilateral both in the lengths of sides and in sector angles. We leave it to the reader to verify the validity of these assertions. We note only that it suffices to make repeated use of Theorem 11 of Chapter VI on the variations of the angles when the triangles are swung onto the plane.

If now, fixing \( \varepsilon_1 \), we decrease \( \varepsilon_2 \) and \( r \), then the curve \( \ldots B_{i-1}B_iB_{i+1}\ldots \) with the adjunction of polygonal pieces enveloping the endpoints of \( L \) bounds a polygon \( P \), with its area \( \sigma_0(P) \) tending to zero as \( \varepsilon_2, r \to 0 \). Therefore, from Lemma 5, it follows that \( \sigma(L) = 0 \).

3. The equation \( \sigma(\gamma) = 0 \) for any polygon or geodesic \( \gamma \) follows from the equation \( \sigma(L) = 0 \) for shortest arcs \( L \) and the additivity of \( \sigma \).

This completes the proof of Theorem 5,

**Theorem 6.** For polygons \( P \) the area \( \sigma(P) \) in the sense of definition (9) coincides with the earlier definition of the area \( \sigma_0(P) \) by equation (5):
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(12) \( \sigma(P) = \sigma_0(P) \).

**Proof.** By Theorem 5, for any \( \varepsilon > 0 \) the boundary of \( P \) may be enclosed in an open set \( g \) for which \( \sigma(g) < \varepsilon \). Suppose that \( P_- \) is the interior region of \( P \), and put \( G = P_- + g \). From (11) and (10) we have

\[
\sigma(P_-) \leq \sigma_0(P) \leq \sigma(G),
\]

but

\[
\sigma(G) - \sigma(P_-) = \sigma(G - P_-) \leq \sigma(g) < \varepsilon;
\]

so that

\[
\sigma(P) = \inf_{G \supset P} \sigma(G) \leq \sigma_0(P).
\]

But it follows from equation (10) that always \( \sigma(P) \geq \sigma_0(P) \), which along with the preceding proves equation (12).

It follows from Theorem 6 that defining the area of a set using (9) does not in the case of polygons contradict the earlier definition (5). Therefore we shall not henceforth distinguish the areas \( \sigma_0 \) and \( \sigma \).

5. **Supplementary remarks.**

**Theorem 7. If the set \( M \) has interior points, then \( \sigma(M) > 0 \).**

**Proof.** Choose in \( M \) an interior point \( O \) with a nonzero complete angle. Draw from \( O \) two shortest arcs, forming some angle \( \alpha > 0 \). We mark points \( X, Y \) at distances \( r \) from \( O \) on these shortest arcs. The triangle \( T = OXY \) on being swung onto the plane gives a triangle \( T_0 \) with area \( s(T_0) = (r^2/2) \sin \alpha_0 \). By Theorem 3,

\[
\sigma(T) \geq s(T_0) - \frac{1}{2} \tilde{\omega}^-(T)d^2(T).
\]

As \( r \to 0 \) we have \( s(T_0) \approx (r^2/2) \sin \alpha \), \( d(T) \leq Cr \), \( \tilde{\omega}^- \to 0 \). Therefore for sufficiently small \( r \) we obtain \( \sigma(T) > 0 \). This proves Theorem 7.

**Remarks.** 1) In the case of a repeatedly differentiable two-dimensional Riemannian surface the area in the sense of Definition (9) coincides on measurable sets with the area in the sense of the usual definitions of differential geometry. We shall not give the proof of this quite natural assertion.

2) Definition (9) was constructed in complete correspondence with the general scheme of § 1, Chapter V. The role of the elementary sets \( t \) was played by the triangle, the role of the function \( \phi(t) \) by the area \( s(t_0) \) of the triangle developed onto the plane, and the role of the resulting measure \( \mu \) by the area \( \sigma \).
In this case the intermediate figures $P$ (polygons), which figured in the general scheme of §1 of Chapter V, play an essential role. For the curvature the situation was different. There the curvature of open sets could be defined directly without the intermediate sets $P$. The following example shows that this is not what happens in the case of area, i.e., we never define the area of open sets by the equation

$$\sigma^*(G) = \lim \sup_{\max \, d(t_i) \to 0} \sum s(t_i^0),$$

where $\{t_i\}$ are finite systems of nonoverlapping triangles $t, \subset G$.

**Example.** Consider a sequence of plane equilateral triangles decreasing unboundedly in measure, whose total area is infinite. We enclose each such triangle $t_i^0$ in three isosceles triangles, cross-hatched in Figure 100. Suppose that the altitude of these triangles decreases so fast that the total area of the cross-hatched triangles is finite, taken together.

We suppose that in an open plane square we have made cuts $X_n, Y_n$ converging to the edge. Each such triple of triangles $AC'B, BA'C, CB'A$ is excised from the plane and the segments $AB'', AC'', BC'', BA'', CA'', CB''$ are pasted pairwise together. The resulting figure is pasted along the exterior contour $AC'B'A'C'B'A'$ to the corresponding cut $X_n, Y_n$.

There results a locally polyhedral surface with an infinite number of boundaries condensing towards the edge. The area introduced above for it, if it is the ordinary area on the plane portions and is completely additive, must turn out to be finite in the large. At the same time the Definition (13) would lead us in this case to an infinite area, since for any $\varepsilon > 0$ it is possible to find a collection of triangles $t_i$ with diameters less than $\varepsilon$ for which the sum $\sum s(t_i^0)$ will be larger than any number given in advance.

3) The definition of area adopted by us is based on a specific construction. But noting that our space is metric, we could have defined the area as the Hausdorff two-dimensional measure $H^2$. In a two-dimensional manifold of bounded curvature these measures will always coincide on Borel sets: $H^2 = \sigma$. This was proved in [70].
3. Convergence of areas.

6. Areas of converging polygons. Suppose that on a set $M$ there are given a metric $\rho$ and a converging sequence of metrics $\rho_n \to \rho$, with all the metrics $\rho_n$ defining the same topology on $M$ and converting $M$ into two-dimensional manifolds $R_n$ of bounded curvature. Suppose that the absolute curvatures $\Omega(R_n)$ are bounded throughout. In this case the limit metric $\rho$ in its turn defines a two-dimensional manifold of bounded curvature. Suppose further that polygons $P_n$ are given in the metrics $\rho_n$, which in the metric $\rho$ are figures which may be considered as curvilinear “polygons” $P'_n$, distinguishing in them the original vertices and sides of the polygons $P_n$. Suppose finally that in the metric $\rho$ the figures $P'_n$ together with vertices and sides converge to some polygon $P$. Then the areas $\sigma_n(P_n)$ of the polygons $P_n$ in the spaces $R_n$ converge to the area $\sigma(P)$ in the space $R$.

**Theorem 8.** If in the converging metrics $\rho_n \to \rho$ the polygons $P_n$ converge to $P$ in the sense of Definition 2 of Chapter VII, then their areas converge:

(14) \[ \sigma_n(P_n) \to \sigma(P). \]

Here, as in the case of reduced triangles, we do not in general include polygons “with tails,” i.e., with sides which coincide on some initial segment on issuing from the common vertex. But in the proof, for simplicity, we suppose that all the $P_n$ and $P$ are bounded only by simple curves. The more general case requires some evident stipulations, the general course of the proof being preserved.

**Proof.** Suppose that relation (14) is not so. Then there exists a sequence $P_n \to P$ for which

(15) \[ |\sigma_n(P_n) - \sigma(P)| \leq \varepsilon_0 > 0. \]

We decompose $P$ into triangles $T'$ with intrinsic diameters not larger then $d'$. For any finite system of points on the sides of $T'$ there will be two types of points, those lying on the boundary of $P$ and those lying inside $P$. To points of the first type there correspond (with respect to the parameter on the converging boundary curves) points on the boundaries of $P_n$, whereas points of the second type for sufficiently large $n$ all fall inside $P_n$ and may therefore be put into correspondence with the same points but already in $P_n$. Thus to each point of a finite collection of points in $P$ there corresponds a definite point in each $P_n$. We include in the chosen collection the following three types of points: 1. All vertices $A$ of the
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

trinagles $T'$. 2. Points $A'$ close to $A$, one on each of the sides of $T'$ issuing from $A$. If we join the points $A_n A'_n$ corresponding to $A, A'$ in $P_n$ by shortest arcs in $R_n$, passing them along the side of $P$ if the corresponding points $A, A'$ lie on one side of $P$, and otherwise drawing them in such a way that there will be no superfluous intersections of shortest arcs, then because of the closeness of the selected points $A'$ to the corresponding point $A$ we can arrange that the shortest arcs $\overline{AA'}$ in $R$ which are limiting for $A_n A'_n$ (convergence may be assured by an appropriate choice of a subsequence of numbers $n$) will be situated around each point $A$ in the same order as the shortest arcs $AA'$. Here it is of course not excluded that these shortest arcs acquire common initial pieces close to the points $A$. 3. Finally, on the pieces $A'A'$ of each side of each of the triangles $T'$ we choose a finite net of points $B$ sufficiently close to one another that, successively joining the points in $P_n$ corresponding to them, we obtain polygons in $P_n$ which converge (convergence being guaranteed by the choice of a subsequence of the numbers $n$) to shortest arcs $\overline{A'A}$ which along with the shortest arcs $\overline{AA'}$ obtained earlier form, possibly a new net $T$, not coinciding with the sides of $T'$, which decomposes $P$ into triangles $T$. The $T$ have the same vertices $A$ as the $T'$, and form a net of the same structure and each side of $T$ coincides with the previous side of $T'$ at least at the points $AA'BB\cdots BA'A$. To each triangle $T$ in the polygon $P_n$ (more precisely in a subsequence selected from them) there corresponds a polygon $Q_n$, bounded by three polygons $A_n A'_n B_n B'_n \cdots B'_n A'_n A_n$, each of which converges in the metric $\rho$ to the corresponding side of $T$. Thus the net of triangles $T'$ is replaced by a net of triangles $T$, limiting for the polygons $Q_n$ of subdivisions of $P_n$. The maximum $d_1$ of the intrinsic diameters of the triangles $T$ will be arbitrarily small along with $d'$. We draw in $Q_n$, without superfluous mutual intersections, three relative shortest arcs, joining vertices which correspond to the vertices of $T$. These shortest arcs decompose $Q_n$ into triangles $T_n$ and three "crusts" $\Gamma_n$.

By Theorem 4 of Chapter VII, the metrics $\rho_{Q_n}$ induced by the excision of the polygons $Q_n$ in $R_n$ converge to a metric $\rho_T$ induced by the excision of $T$ in $R$. Therefore we will have the following convergences:

1) the intrinsic diameters $d_{Q_n}\to d_T$;

2) the lengths of the sides of $T_n$ converge to the lengths of the sides of $T$, and therefore also the areas of the plane triangles with sides of the same length also converge: $s(T_n)\to s(T)$;

3) the lengths of the "sides" of the polygons $Q_n$, by the construction,
also converge to the lengths of the sides of $T$, so that if any of the three crusts $\Gamma_n$ is decomposed into triangles by relative shortest arcs in $Q_n$ drawn from the vertices of $\Gamma_n$ and these triangles developed onto the plane, then from them one may make up a plane polygon $\Gamma_0$ whose area $s_n(\Gamma_0) \to 0$ with increasing $n$.

Now we denote by $\Omega$ the upper bound of all $\Omega(P)$, $\Omega(P_n)$, $\Omega(P^n)$. Then on the basis of Theorem 3, in which we may because of Theorem 6 regard $\sigma_0$ as simple area, we will have

$$
|\sigma_n(P_n) - \sigma(P)| = \left| \sum_{r_n} \sigma_n(\Gamma_n) - \sum_T (T) \right| = \left| \sum_{r_n} \sigma_n(\Gamma_n) + \sum_{r_n} \sigma_n(T_n) - \sum_T (T) \right|
$$

$$
\leq \left| \sum_{r_n} \sigma_n(\Gamma_n) - \sum_{r_n} s(T_n^0) \right| + \left| \sum_{r_n} s(T_n^0) \right| + \left| \sum_{r_n} \sigma_n(T_n) - \sum_T (T) \right|
$$

(16)$$
\leq \frac{1}{2} \Omega \max d^2_{\gamma} + \epsilon + \frac{1}{2} \Omega \max d^2_{\gamma} + \epsilon + \frac{1}{2} \Omega \max d^2_{\gamma}.
$$

For sufficiently small $d'$ and sufficiently large $n$ the right side of inequality (16) becomes less than $\epsilon_0$, leading to a contradiction with the supposition (15). This proves Theorem 8.

7. Weak convergence of areas.

**Theorem 9.** If on the set $M$ there are given metrics $\rho_n$ with absolute curvatures bounded uniformly, the $\rho_n$ defining on $M$ the same topology and uniformly converging to a metric $\rho$, then the areas $\sigma_n$ defined in the metrics $\rho_n$ converge locally weakly as set functions to the area $\sigma$ in the metric $\rho$:

$$
\sigma_n \overset{\text{l.w.}}{\to} \sigma.
$$

It is also sufficient to have local boundedness of the $\Omega_{\rho_n}$ and local uniform convergence $\rho_n \to \rho$.

**Corollary.** Under the conditions of Theorem 9, on each open set $G$ whose boundary has zero area in the metric $\rho$, we have the usual convergence

$$
\sigma_n(G) \to \sigma(G).
$$

**Proof.** Using Kolmogorov’s test, Lemma 16 of Chapter VII, for the proof of weak convergence it suffices to show that for any $F_0 \subset G_0$, where $F_0$ is a closed set and $G_0$ an open set with a compact closure, the relation

$$
\lim_{n \to \infty} \inf_{F_0 \subset G \subset G_0} |\sigma_n(G) - \sigma(G)| = 0
$$

holds. But since for such $F_0$ and $G_0$ there is always an open set $G$ satisfying the conditions
it suffices to prove the convergence (18) formulated above in order to show the validity of Theorem 9.

Thus suppose that \( \sigma(\text{Fr. } G) = 0 \). Taking an arbitrarily small \( \varepsilon > 0 \), we choose two polygons \( p, P \) such that the conditions

\[ p \subset G \subset \bar{G} \subset P, \quad \sigma(G) - \varepsilon < \sigma(p) < \sigma(P) < \sigma(G) + \varepsilon \]

are satisfied. Such a choice for a set \( G \) with a compact closure is possible since

\[ \sigma(G) = \sup_{p \subset \bar{G}} \sigma(p), \quad \sigma(G) = \sigma(\bar{G}) = \inf_{P \supset \bar{G}} \sigma(P). \]

We choose further polygons \( q, Q \) such that

\[ p \subset q_+ \subset q \subset G \subset \bar{G} \subset Q_- \subset Q \subset P_- \subset P. \]

On the boundaries of \( q, Q \) we choose a sufficiently dense net of points and, successively joining them by shortest arcs in the metric \( \rho_n \) without superfluous intersections, we obtain polygons \( q_n, Q_n \). This is done analogously to the proof of Theorem 8, in the construction of the polygons \( Q_n \).

Then we choose a subsequence of numbers \( n_1 \) such that on this subsequence the polygons \( q_{n_1}, Q_{n_1} \) converge to some limiting polygons \( q', Q' \), and such that the quantities \( \sigma_n(q_{n_1}) \) and \( \sigma_n(Q_{n_1}) \) should have limits. Then, using Theorem 8 for converging polygons, we have

\[ \sigma(G) - \varepsilon < \sigma(p) \leq \sigma(q') = \lim \sigma_n(q_{n_1}) \leq \lim \sup \sigma_n(q_{n_1}) \leq \lim \inf \sigma_n(G) \]

\[ \leq \lim \sup \sigma_n(G) \leq \lim \sup \sigma_n(\bar{G}) \leq \lim \inf \sigma_n(Q_{n_1}) \]

\[ \leq \lim \sigma_n(Q_{n_1}) = \sigma(\bar{G}) \leq \sigma(P) < \sigma(G) + \varepsilon, \]

i.e.,

\[ \sigma(G) - \varepsilon < \lim \inf \sigma_n(G) \leq \lim \sup \sigma_n(G) < \sigma(G) + \varepsilon, \]

which, in view of the arbitrary smallness of \( \varepsilon > 0 \), proves relation (18) and with it Theorem 9.

The following example shows that under the conditions of Theorem 9 there is generally speaking only weak convergence \( \sigma_n \to \sigma \), and not convergence of these functions on each Borel set.

**Example.** Consider a square \( K \) with side 1. On one side of the square we remove a center segment of length 1/4. In the remaining two pieces of the side we again remove from each piece an open segment of length

\(^1\) Here we are using a strengthened formulation of Theorem 8, since the polygons \( q_{n_1} \to q' \) and \( Q_{n_1} \to Q' \) may have "tails" close to the vertices.
1/4^2 \cdot 1/2. At the third stage we remove from each of the $2^2$ remaining pieces an open middle segment of length $1/4^3 \cdot 1/2^2$, and so forth, on the $n$th step removing from each of the $2^{n-1}$ remaining pieces an open middle segment of length $1/4^n \cdot 1/2^{n-1}$. Over each removed segment we construct a strip through the entire square. All of these open strips, taken together, form a set $G$ with area

$$\sigma(G) = \frac{1}{4} + \frac{1}{4^2} + \cdots = \frac{1}{3}.$$ 

Moreover, consider a sequence $K_1, K_2, \cdots$ of such squares. For each square we repeat a similar construction with the differences that in the square $K_n$ the removal of strips up to the $n$th step is done in the same way as for the square $K$, and, beginning with the $(n+1)$th step we successively remove middle pieces equal to one half of the pieces from which they are removed. The sum of the strips removed from $K_n$ forms a set $G_n$, but this time all the areas $\sigma(G_n) = 1$.

Now suppose that $\phi_n$ is a mapping of the square $K$ onto $K_n$ which consists of mapping each strip of the set $G$ (in case of necessity by uniform expansion) onto the corresponding strip of the set $G_n$, and then this mapping is extended onto all of $K$ by continuity.

For each $n$ we can introduce a metric $\rho_n$ on $K$, defining $\rho_n(X,Y)$ as the distance between the points $\phi_n(X)$ and $\phi_n(Y)$ in $K_n$. All the metrics $\rho_n$ are metrics of bounded curvature, since they are metrics of the plane squares $K_n$. Moreover, on all of $K$ there is uniform convergence $\rho_n \to \rho$, where $\rho$ is the usual metric in $K$. At the same time

$$1 = \sigma_{\rho_n}(G) \to \sigma_{\rho}(G) = \frac{1}{3}.$$
Chapter IX

Curves with Rotation of Bounded Variation

1. Variation of the rotation. As in the preceding chapters, we will always be considering a space which is a two-dimensional manifold of bounded curvature.

1. Arc with rotation of bounded variation. An open simple arc $\mathcal{A}$ is a curve obtained from the simple arc $\overline{\mathcal{A}}$ by excluding its endpoints.

**Definition 1.** We shall say that the open simple arc $\mathcal{A}$ has a rotation of bounded variation if the following three conditions are satisfied:

1) on the arc $\overline{\mathcal{A}}$ there are no points with zero complete angle $\theta = 0$;
2) at each interior point the arc $\mathcal{A}$ has a definite direction of arrival at that point and departure from that point;
3) for all finite collections of points $A_i (i = 1, \cdots, n)$ following one another along $\mathcal{A}$ the sum

\[
\sum_{i=1}^{n-1} |\tau_r(A_iA_{i+1})| + \sum_{i=2}^{n-1} |\tau_l(A_i)| \leq N(\mathcal{A})
\]

remains bounded.

Here the right rotation $\tau_r(A_iA_{i+1})$ of each open piece $A_iA_{i+1}$ of the curve $\mathcal{A}$ certainly exists in view of condition 1 and Theorem 2 of Chapter VI. By the right rotation $\tau_r(A_i)$ at the point $A_i$ is meant the quantity $\pi - \bar{\alpha}_i$, where $\bar{\alpha}_i$ is the sector angle at the vertex $A_i$ formed by the branches of $\mathcal{A}$ and lying to the right of $\mathcal{A}$; this sector angle exists because of condition 2.

**Definition 2.** The least upper bound of the sums on the left side of inequality (1) with respect to all possible systems of points $\{A_i\}$ is called the variation of the right rotation of the open arc $\mathcal{A}$, and is denoted by $\sigma_r(\mathcal{A})$.

Analogously one defines the left rotation $\tau_l(\mathcal{A})$ and its variation $\sigma_l(\mathcal{A})$.

Since $\mathcal{A}$ is a piece of the compact set $\overline{\mathcal{A}}$, the absolute curvature $\Omega(\mathcal{A})$ is finite. Moreover, for the left and right rotations of the pieces and points of the curve $\mathcal{A}$ we have from Theorem 6 of Chapter VI the relations

\[
\tau_r(A_iA_{i+1}) + \tau_l(A_iA_{i+1}) = \omega(A_iA_{i+1}), \quad \tau_r(A_i) + \tau_l(A_i) = \omega(A_i).
\]

From (2) follows
Lemma 1. Under the conditions of Definition 1, not only the sums in the inequality (1) but also the analogous sums for the left and right rotations are bounded, and the following relation holds:

\[ |\sigma_r(\mathcal{L}) - \sigma_l(\mathcal{L})| \leq \Omega(\mathcal{L}). \]

Remark. A curve homeomorphic to the open disc generally speaking may fail to be a simple open arc. It may turn out that one cannot assign endpoints to it. On Figure 101 we depict such a geodesic on a regular surface, infinite on one side. In Figure 102 we show an open square with an infinite series of paste-ins in the form of twice-covered triangles. The segment \( \mathcal{L} \) running in this square has no endpoints within the limits of the open square. In this case \( \Omega(\mathcal{L}) = \infty \), and if one introduces the quantities \( \sigma_1(\mathcal{L}), \sigma_l(\mathcal{L}) \), only one of them will turn out to be finite.

2. Rotation, its positive and negative parts and its variation. Select on \( \mathcal{L} \) a system \( \{t\} \) of sets \( t \), to which we refer all the closed portions, i.e., all partial arcs on \( \mathcal{L} \) and all one-point sets on \( \mathcal{L} \). Two sets \( t_1, t_2 \in \{t\} \) are taken to be nonoverlapping if they are either two arcs without common interior points or two noncoincident points, or an arc and a point not lying inside it. By \( T \) we shall mean a finite system of pairwise nonoverlapping \( t \).

On the sets \( t \) there is defined the function \( \tau_r(t_-) \), where by \( t_- \) we mean \( t \) itself if \( t \) is a point and the interior portion of \( t \) if \( t \) is an arc. Therefore on \( t \) we have defined the functions:

\[ \phi^+(t) = \max\{0, \tau_r(t_-)\}, \]

\[ \phi^-(t) = \max\{0, -\tau_r(t_-)\}, \]

\[ |\phi|(t) = |\tau_r(t_-)|. \]
**Definition 3.** By \( G \) we denote open sets on the simple arc \( \mathcal{L} \). We then have on sets \( M \subset \mathcal{L} \) the nonnegative set functions

\[
\tau^+ (M) = \inf \sup \sum \phi^+(t), \quad G \supset M, T \subset G, t \in T
\]

\[
\tau^- (M) = \inf \sup \sum \phi^-(t), \quad G \supset M, T \subset G, t \in T
\]

\[
\sigma_r (M) = \inf \sup \sum | \phi | (t), \quad G \supset M, T \subset G, t \in T
\]

and the function

\[
\tau_r (M) = \tau^+ (M) - \tau^- (M).
\]

By the usual methods of measure theory we may prove the following two lemmas.

**Lemma 2.** Let \( M \) be an open segment \( AB \) on \( \mathcal{L} \). Then the values of \( \tau_r (M) \) and \( \sigma_r (M) \) defined by (8) and (7) coincide respectively with the original value of the rotation, \( \tau_r (AB) \), and the value \( \sigma_r (AB) \) in the sense of Definitions 1 and 2. If \( M \) is a one-point set \( A \), then \( \tau_r (M) \) is equal to the rotation \( \tau_r (A) \), and

\[
\tau^+ (M) = \max \{ 0, \tau_r (A) \}, \\
\tau^- (M) = \max \{ 0, - \tau_r (A) \}, \\
\sigma_r (M) = | \tau_r (A) |. 
\]

This lemma justifies the continuation in Definition 3 of the rotations \( \tau_r \) and \( \sigma_r \) required earlier. We keep for \( \tau_r, \tau^+_r, \tau^-_r, \sigma_r \) the expressions: right rotation of the curve, its positive and negative parts and the variation of the right rotation, understanding them in what follows as set functions on the curve.

**Lemma 3.**

\[
\sigma_r (M) = \tau^+_r (M) + \tau^-_r (M).
\]

**Theorem 1.** The functions \( \tau^+_r, \tau^-_r, \sigma_r \) are completely additive regular Carathéodory measures on the Borel sets of the arc \( \mathcal{L} \).

**Proof.** From Theorem 4, Chapter VI, on the additivity of the rotation, it follows that on decomposing the arc into pieces \( t_i \), counting among the \( t_i \) the successive intervals and division points, one always has the inequalities

\[\text{On } \mathcal{L} \text{ we may introduce a strictly monotone parameter and take these functions to be given on sets of values of the parameter, i.e. in a metric space, and therefore speak of the Carathéodory measure.}\]
Therefore each of the functions $\phi^+, \phi^-, |\phi|$ satisfies the conditions of Theorem 3 of Chapter IV, from which the validity of the present theorem follows.

**Corollary.** The function $\tau_r(M)$ is also completely additive on the Borel sets $M$ of the arc $\mathcal{A}$.

3. **Direction at the ends of an arc.**

**Theorem 2.** An arc $\mathcal{A}$ with rotation of bounded variation has at each of its endpoints a definite direction.

**Proof.** By definition of a curve with rotation of bounded variation, the complete angle $\theta > 0$ at the endpoint $A$ of the arc $\mathcal{A}$. We shall show that for any shortest arc $M$ issuing from $A$ the curve $\mathcal{A}$ close to $A$ may, if it intersects $M$ repeatedly, envelop $A$ only a bounded number of times.

To this end we consider a polygonal neighborhood $P$ of the point $A$ so small that $\Omega(P-A) < \theta/2$ and the shortest arc $M$ leaves the limits of $P$. Suppose that $\mathcal{A}$ envelops the point $A$ within $P m$ times. In Figure 103 we depict two circuits: $A_{k-1}A_k$ and $A_kA_{k+1}$. Choose $m$ exemplars of the polygon $P$. Cutting each of them along $M$ and pasting the successive edges of the cuts together, we obtain an $m$-sheeted surface with branch
point $A$. Now consider the curve $A_1A_2$ on this multi-sheeted surface, taking the first circuit $A_1A_2$ to be accomplished on the first sheet of the surface, the second on the second and so forth, finally the last, $A_mA_{m+1}$ on the $m$th. Then the segments $AA_1, AA_{m+1}$ of the shortest arc $M$ and the piece $A_1A_{m+1}$ of the arc $\mathcal{S}$ excise from the multi-sheeted surface a region homeomorphic to the disc, as depicted on Figure 104. With the notations for the rotations, curvatures and sector angles indicated in Figure 104 we have, because of the connection between the curvature and the rotation of the boundary:

$$(\pi - \alpha) + \tau + (\pi - \beta) + \tau_1 + (\pi - m\theta) + \tau_2 = 2\pi - \omega.$$ 

Therefore noting that

$$\alpha, \beta \geq 0, \quad \tau_1, \tau_2 \leq 0, \quad \omega \leq m\Omega,$$

we obtain

$$(10) \quad m \leq \frac{\pi + \tau + \Omega}{\theta - \Omega} \leq \frac{\pi + \sigma + \theta/2}{\theta/2},$$

where $\sigma$ is the variation of the rotation of $\mathcal{S}$.

Thus the above assertion is proved.

Now suppose that $\mathcal{S}$ does not have a definite direction at $A$. Then on moving the point $X$ along $\mathcal{S}$ towards $A$ the direction of the shortest arc $AX$ (Lemma 1, Chapter VI) does not stabilize. Thus there are two shortest arcs $M_i$ and $M_2$ issuing from $A$, which form with one another an angle $\phi > 0$, and the arc $\mathcal{S}$ nevertheless intersects each of the shortest arcs $M_i, M_2$ an infinite number of times in an arbitrarily small neighborhood of the point $A$.

From what was proved above, near $A$ there are positions when $\mathcal{S}$ successively intersects one of the curves $M_i, M_2$, then the other and again the first, without in doing so enveloping the point $A$ (Figure 105). In this case, with the notations of Figure 105, $\alpha \geq \min [\bar{\alpha}, \bar{\alpha}', \pi] = \phi > 0$. We may suppose that the construction was carried out in a neighborhood of $A$ in which the rotations of $M_i, M_2, \mathcal{S}$ and the curvatures of the region, with the point $A$ removed, are very small in comparison with $\phi$. (The rotation of $\mathcal{S}$ near the endpoint is small because of the complete additivity of the rotation.)

Then from the consideration of the “two-gon” $BE$ (Figure 105) we conclude that the sector $\tilde{\beta}$ is small, and on considering the “two-gon” $CD$ that the sector $\tilde{\delta}$ is small and (taking account of the smallness of the absolute curvature at the point $C$) that the sector $\tilde{\gamma}$ is close to $\pi$. Finally,
considering the "triangle" $ABC$, which has a small curvature and sides whose rotations are small, leads us to a contradiction with the fact that $\alpha + \beta + \gamma$ is appreciably (nearly by $\phi$) larger than $\pi$.

This proves Theorem 2.

Figure 105. Figure 106.

Now consider any simple arc $\mathcal{L}$ with a rotation of bounded variation. According to Theorem 2, $\mathcal{L}$ has directions at the endpoints $A, B$. If we may pass shortest arcs from these endpoints which do not intersect $\mathcal{L}$ except as at the original points, then the rotation $\tau(AB)$ is defined for the entire arc $\mathcal{L}$. Earlier we had the rotation $\tau(A_nB_n)$ on each shortened piece of the arc $\mathcal{L}$ (Figure 106). Under these conditions we have:

**Lemma 4.**

\[ \tau(AB) = \lim_{A_n \to A, B_n \to B} \tau(A_nB_n). \]  

**Proof.** Because of the definition of rotation it suffices to show that for $A_n, B_n$ close to $A, B$ and polygons $L_0, L_n$ close to $\mathcal{L}(A, B), \mathcal{L}(A_n, B_n)$ (Figure 106), the values

\[ \tau_0 + \alpha + \tilde{\beta}, \quad \tau_n + \alpha_n + \tilde{\beta}_n, \]

are close. But we have
\[ \tau_n + (4\pi - \alpha - \tilde{\alpha}' - \tilde{\beta}' - \tilde{\beta}) + \tau_0 + \tau(AA_n) + \tau(BB_n) = 2\pi - \omega(U), \]
\[ \tau_0 + \tau_0' = \omega_0(L_0 - A - B), \]
\[ |\tilde{\alpha}_n + \tilde{\alpha}'_n - \pi| = |\tau(A_n)|, \]
\[ |\tilde{\beta}_n + \tilde{\beta}'_n - \pi| = |\tau(B_n)|. \]

From these equations, taking account of the smallness of the quantities \( \omega(U), \omega(L_0 - A - B), \tau(A_n), \tau(AA_n), \tau(BB_n) \), we find that the quantities \( (12) \) are close to each other and thus that Lemma 4 is valid.

2. Approximation by polygonal curves.

4. Possibility of one-sided approximation by polygonal curves.

Theorem 3. If \( \mathcal{L} \) is a simple arc with rotation of bounded variation in the sense of Definition 1, then there exists a sequence of simple polygonal curves converging from the right to \( \mathcal{L} \) and with rotations\(^2\) whose variations have an upper limit not exceeding \( \sigma_r(\mathcal{L}) \). Further we may guarantee that these polygons have common endpoints with \( \mathcal{L} \) and form at the endpoints small angles with the curve \( \mathcal{L} \).

Proof. 1. Choose an arbitrary \( \varepsilon > 0 \) and draw a shortest arc \( L \) joining the endpoints \( M \) and \( N \) of the arc \( \mathcal{L} \), passing to the right of \( \mathcal{L} \) and forming with \( \mathcal{L} \) at its endpoints angles less than \( \varepsilon \), and bounding along with \( \mathcal{L} \) a region which is homeomorphic to the closed plane disc and has an absolute curvature, on deletion of the points of the arc \( \mathcal{L} \) and its endpoints, also less than \( \varepsilon \). (All of this may be accomplished, since \( \mathcal{L} \) has a definite direction at its endpoints and at these endpoints \( \theta \neq 0 \).)

2. Consider moreover a sequence of other polygonal curves \( L_k \), joining the points \( M \) and \( N \), passing between \( \mathcal{L} \) and \( L \) and such that each successive polygon \( L_k \) passes to the left of the preceding \( L_{k-1} \) and the broken curves \( L_k \to \mathcal{L} \) as \( k \to \infty \). The pair \( L_k \) and \( L \), for each \( k \), bounds a closed polygon \( P_k \) homeomorphic to the disc.

Draw in \( P_k \) the shortest curve \( l_k \) joining the points \( M \) and \( N \) and in addition the “rightmost”, i.e., the “closest” to \( L \) of such curves if there are several. As is known, a shortest arc in a polygon will be a geodesic polygon each turning point of which coincides with one of the reentrant vertices of the polygon. In addition, if \( l_k \) has a turning point at the vertex \( Y \) of the polygon \( L_k \), then the right rotation of \( l_k \) and moreover of \( L_k \) at

\(^2\) These polygonal curves run in regions of infinitely decreasing absolute curvature and therefore it does not matter whether we are speaking of the variation of their left or right rotations.
that vertex is *a fortiori* nonpositive, and if the turning point takes place at a vertex $X$ of the polygonal curve $L$, then the left rotations of $l_k$ and $L$ at the vertex $X$ are certainly nonpositive.

From the construction of the broken curves $l_k$ it follows at once that each successive one of them has no points to the right of its predecessor. Therefore it follows that if at least one of the curves $l_k$ fails to pass through some vertex $X$ of the polygonal curve $L$, then every following $l_{k+p}$ also fails to pass through $X$. Thus, one finds only a definite number of vertices $M = X_1, X_2, \cdots, X_n, X_{n+1} = N$ of the polygon $L$ through each of which there pass infinitely many polygons $l_k$. Dropping a finite number of broken curves $l_k$, we may suppose that all the $l_k$ are stretched on the indicated vertices of $L$ and not on the other vertices of $L$.

3. The polygons $l_k$ lie in a compact region $D$ bounded by the curves $\mathcal{Q}$ and $L$. Moreover their lengths are bounded uniformly, since they do not increase, and the length of $l_1$ does not exceed the finite length of $L$. Therefore it is possible to find a sequence of $k$ for which the $l_k$ converge to some limit curve $l$. Further we suppose that we are dealing only with this subsequence. From the minimal properties of the $l_k$ it follows that $l$ is a simple arc joining the points $M$ and $N$ and passing through the region $D$. As with all the $l_k$, the curve $l$ will pass through $X_1, \cdots, X_{n+1}$.

4. If on some portion $X_i, X_{i+1}$ of the limit curve $l$ there are no points in common with $\mathcal{Q}$, then all the $l_k$, not passing to the left of $l$, are not spanned on the broken curves $L_k$ after a certain $k$, and are segments of geodesics on the portion $X_iX_{i+1}$.

Now we consider a portion $X_iX_{i+1}$ on which $l$ has common points with $\mathcal{Q}$. Among these common points there is a first $Z_1$ and a last $Z_2$, in the order of succession of points along $\mathcal{Q}$ and $l$. We do not exclude the possibility $Z_1 = Z_2$. We note that even if on the piece in question certain $l_k$ are not spanned on $L_k$, we may alter $L_k$, moving back on the side of $l_k$, in such a way that on the piece $X_iX_{i+1}$ each polygon $l_k$ is spanned at certain points on $L_k$ and the first and last of these points, $Y_1$ and $Y_2$, converge respectively to $Z_1$ and $Z_2$. Everything we have said is sketched in Figure 107.

The pieces $X_iY_1$ and $Y_2X_{i+1}$ are relative shortest arcs and therefore geodesics. By the same principle their limits $X_iZ_1$ and $Z_2X_{i+1}$ are also geodesics. Therefore in particular the latter curves have definite directions at their endpoints. Moreover, from the convergence $X_iY_1 \to X_iZ_1$ and $Y_2X_{i+1} \to Z_2X_{i+1}$ it follows that as $k \to \infty$ the angles $\angle Z_1X_iY_1$ and $\angle Z_2X_{i+1}Y_2$ tend to zero.
5. We note further that the piece $\overline{Z_iZ_2}$ of the limit curve $l$ also has definite directions at its endpoints, for example at $Z_2$. This follows from the fact that as the point $T \in \overline{Z_iZ_2}$ approaches $Z_2$ the direction of the shortest arc $T_2Z_2$, drawn so that it has no superfluous intersections with the earlier drawn shortest arcs $Z_2T_1$, may only swing monotonically to the right around $Z_2$. Indeed, if there were some shortest arc $T_2Z_2$ which passed to the left of a preceding shortest arc $T_1Z_2$, then as $k \to \infty$ the polygons $l_k$, converging from the right to $\overline{Z_iZ_2}$, would have to intersect the shortest arc $T_1Z_2$ as they approach $T_2$. But then the polygon $l_k$ may either be shortened or replaced by one further left, which contradicts the choice of $l_k$.

6. Suppose finally that $k$ is so large that all the angles $\angle Z_1X_iY_1 < \varepsilon/2n$ and $\angle Z_2X_{i+1}Y_2 < \varepsilon/2n$.

Denoting by $d_1^k$ the region bounded by the curves $X_iZ_1Z_2X_{i+1}$ and the piece $X_iX_{i+1}$ of the curve $l_k$, with the exception of the curve $X_iZ_1Z_2X_{i+1}$ itself. From the connection between the rotation of the boundary of this region and the curvature of the region it follows that

$$\tau_r(X_i\overline{Z_1Z_2X_{i+1}}) = \tau_r(X_iX_{i+1}) + \angle Z_1X_iY_1 + \angle Z_2X_{i+1}Y_2 - \omega(d_1^k),$$

so that

(13)  
$$\tau_r(X_i\overline{Z_1Z_2X_{i+1}}) \leq \tau_r(X_iX_{i+1}) + \frac{\varepsilon}{n} + \Omega(d_1^k).$$

Consider separately the rotation $\tau_r(X_i\overline{Z_1Z_2X_{i+1}})$ on the portions $X_iZ_1$, $Z_1$, $\overline{Z_iZ_2}$, $Z_2$, $Z_2X_{i+1}$. On the open portions $XZ_1$ and $Z_2X_{i+1}$ it is nonpositive and does not exceed $\Omega(X_iZ_1)$ and $\Omega(Z_2X_{i+1})$ respectively in absolute value. On the open portion $\overline{Z_iZ_2}$ the right rotation is connected with the right rotation of the curve $\mathcal{C}$ itself on the corresponding portion $\overline{Z_1Z_2}$ by the relation

(14)  
$$\tau_r(\overline{Z_1Z_2}) = \tau_r(\overline{Z_1Z_2}) - \alpha - \beta + \omega(g_i),$$

where the notations $\alpha$ and $\beta$ are given in Figure 107 and $g_i$ is the region bounded by $\overline{Z_1Z_2}$ and $\overline{Z_2Z_2}$ with the exception of the points of the curve $\overline{Z_1Z_2}$ itself. Finally, at the points $Z_1, Z_2$ the right rotation of the curve

![Figure 107.](image-url)
$X_iZ_1Z_2X_{i+1}$ in question is connected with the right rotation $\tau_r(Z_i)$, $\tau_r(Z_3)$ at the same points on the curve $\mathcal{L}$ by the inequalities

\[
\tau_r(Z_i) \geq \tau_r(\mathcal{L}) + \alpha,
\]

\[
\tau_r(Z_3) \geq \tau_r(\mathcal{L}) + \beta,
\]

since $l$ passes to the right of $\mathcal{L}$.

Inequalities (13)—(15) allow us to conclude that

\[
\tau_r([Z_1Z_2]) < \tau_r(X_iX_{i+1}) + \frac{\varepsilon}{n} + \Omega(d_1^i + g_i + X_iZ_i + Z_2X_{i+1}),
\]

where $[Z_1Z_2]$ is a closed portion of the curve $\mathcal{L}$.

7. We have arrived at the following situation. The right rotations of $\mathcal{L}$ and $l_k$, because of the choice of the original curve $L$, differ by less than $3\varepsilon$. The variation $\sigma_r(l_k)$ is made up of the possible positive rotations at the $X_i$, $i = 2, \ldots, n$, and the sum of the negative rotations, which behave as follows: a) at the points $X_i$ they cannot exceed $\Omega(X_i)$, since the left rotations of the $l_k$ at these points are nonpositive; b) on the pieces $X_iX_{i+1}$, where $l$ was not spanned on $\mathcal{L}$, they cannot exceed $\Omega(X_iX_{i+1})$. On the other pieces, from (16), to the negative rotations $\tau_r(X_iX_{i+1})$ there correspond almost the same or still smaller rotations $\tau_r([Z_iZ_2])$ on the pieces of $\mathcal{L}$.

From all that has been said, recalling that the terms $\varepsilon/n$ cannot be repeated more than $n$ times, and that the absolute curvature $\Omega$ of all the regions, points, and curves in question cannot in sum exceed $\varepsilon$, it already follows, that

\[
\sigma_r(\mathcal{L}) > \sigma(l_k) - 5\varepsilon.
\]

Carrying out this construction as $\varepsilon_m \to 0$, we obtain the required sequence of polygons $l_k(m)$.

Remarks. 1) As is clear from the construction, the vertices of the $l_k$ lie at certain vertices of $L$ and $L_k$. Therefore if desired, it may be arranged that the vertices of $l_k$ lie only at points where $\omega = 0$.

2) As we shall see further on (subsection 11) the variations $\sigma_r(l_k(m))$ of the curves $l_k(m)$ just constructed do in fact converge to $\sigma_r(\mathcal{L}_k)$ as $m \to \infty$.

5. Arc and chord.

Lemma 5. If the shortest arc $l$ and the polygon $L$ bound a region $g$ homeomorphic to the disc, while $\omega^+ + \tau^+_i \leq \pi$, where $\tau^+_i$ is the positive part of the rotation of the polygon $L$ from the side of $g$ and $\omega^+$ the positive part of the curvature of the open region $g$, then for the lengths of $L$ and $l$ we have the inequality
(17) \[ L \cos \frac{\omega^+ + \tau^+}{2} \leq l. \]

**Proof.** Excising the polygonal region \( g \) from the manifold in question and pasting a cylindrical piece to it, we may without loss of generality regard the region \( \bar{g} \) as convex. Decompose \( \bar{g} \) into reduced triangles by shortest arcs drawn from one common endpoint of \( l \) and \( L \) to all the remaining vertices of \( L \). Then construct plane triangles with sides of the same length and adjoin them to one another in the same order as in the region \( \bar{g} \). Since the decrease of the angles of the triangles, from Theorem 11 of Chapter IV, does not in sum exceed \( \omega^+ \), the rotation from the side of \( l' \) of the plane polygon \( L' \) thus formed does not exceed \( \tau^+ + \omega^+ \). To \( L' \) and \( l' \) in the plane we apply estimate (23) from Chapter III, obtaining inequality (17).

**Lemma 6.** Suppose that inside the region \( G \), homeomorphic to the open disc and having \( \omega^+ \) as the positive part of its curvature, there lies a simple arc \( \mathcal{Q} \) with bounded variation \( \sigma_r \) of its rotation on the side of \( G \), and a shortest arc \( l \) joining its endpoints. We suppose that \( \sigma_r + \omega^+ < \pi \). Then the arc \( \mathcal{Q} \) is rectifiable and for the lengths of \( \mathcal{Q} \) and \( l \) we have the inequality

(18) \[ \mathcal{Q} \cos \frac{\sigma_r + \omega^+}{2} \leq l. \]

**Proof.** Take an \( \varepsilon > 0 \) such that \( \sigma_r + \omega^+ + \varepsilon < \pi \). Approximate \( \mathcal{Q} \) in accordance with Theorem 3 by a simple polygon \( L \) lying in \( G \) and having the same endpoints, with a variation of its rotation from each of the sides not exceeding \( \sigma_r + \varepsilon \). If \( L \) and \( l \) have no common points other than the endpoints, we may use estimate (17).

If \( L \) and \( l \) have other points of intersection, we assert that these points follow in the same order on each of the curves \( L \) and \( l \). Indeed, if this were not so, then on some portion of \( L \) and \( l \) there would be formed a two-gon of the type \( AB \) depicted in Figure 108. In it the sector angles \( \tilde{\alpha} \geq 0, \tilde{\beta} \geq \pi \), so that on comparing the rotation of the contour with the curvature of the region contained in it we arrive at a contradiction with the condition \( \sigma_r + \varepsilon + \omega^+ < \pi \).

*Figure 108.*
So, \( L \) and \( l \) can intersect, only forming successively a finite or countable number of nonoverlapping two-gons as depicted in Figure 109. Applying estimate (17) to the sides of \( L_t, l_t \) of each of these two-gons, and taking account of the arbitrary smallness of \( \varepsilon \) and the fact that the length of the limiting curve does not exceed the lower limit of the converging curves, we obtain the assertion of Lemma 6.

\textbf{Remark.} In the proof of Lemma 6 the condition \( \sigma_r + \omega^+(G) < \pi \) was used only in clarification of the relative positions of \( L \) and \( l \). In the final estimate the curvature \( \omega^+(G) \) may be replaced by \( \omega^+(G - M - N) \), where \( M \) and \( N \) are the common endpoints of \( \mathcal{L} \) and \( l \).

It follows from Lemma 6 that if \( \mathcal{L} \) and \( l \) lie in a region \( G \) with a small curvature \( \omega^+(G) \) and if \( \sigma_r(\mathcal{L}) \) is small, then the ratio of the lengths of \( \mathcal{L} \) and \( l \) is close to unity.

\textbf{Theorem 4.} Suppose that from the point \( X_0 \), at which \( \rho \neq 0 \), there issues an arc \( \mathcal{L} = X_t \) \((0 \leq t \leq 1)\) with rotation of bounded variation. Then as \( t \to 0 \) the ratio of the length of the arc \( X_0X_t \) to the chord \( X_0X_t \) tends to unity.

\textbf{Proof.} Make a cut along any shortest arc issuing from \( X_0 \). Pasting together the necessary number of copies of the same space, as was done in the proof of Theorem 2, we transfer our discussion to a space in which \( \omega^+(X_0) < \pi \). Then we choose a neighborhood \( G \) of the point \( X_0 \), homeomorphic to the disc and an initial segment of \( \mathcal{L} \), both so small that in \( G \) the estimate \( \sigma_r(\mathcal{L}) = \omega^+(G) < \pi \) is valid and the shortest arc \( \overline{X_0X_t} \) does not leave \( G \). Then from Lemma 6, taking account of the remark to that lemma, we will have

\[
\frac{X_0X_t \cos \sigma_r(\overline{X_0X_t}) + \omega^+(G - X_0)}{2} < \overline{X_0X_t},
\]

which in view of the smallness of \( \sigma_r(\overline{X_0X_t}) \) and \( \omega^+(G - X_0) \) for small \( G \) and \( t \) proves Theorem 4.

6. \textit{Singularities connected with points where} \( \theta = 0 \).

\textbf{Remarks.} 1) In defining the simple arc \( \mathcal{L} \) with rotation of bounded variation we excluded the possibility of the existence on \( \mathcal{L} \) or at the endpoints of \( \mathcal{L} \) points where \( \theta = 0 \). If we consider an arc satisfying all the conditions of Definition 1 but having one or both endpoints at points where \( \theta = 0 \), then for such a curve Theorem 2 is true in an evident way. The direction at the endpoint of that curve where \( \theta = 0 \) exists because of the
trivial principle that any curve at such a point has a direction. In just
the same way Lemma 4 also remains valid. Therefore to such a curve
one may assign a definite rotation, as the limit of the rotations of extending
arcs, even in the case when it is not possible to draw from the endpoints
of $\mathcal{S}$ shortest arcs which do not intersect $\mathcal{S}$. Analogously we may consider
simple arcs with any finite number of points where $\theta = 0$, with $\tau_r, \tau_r^+, \tau_r^-$,
$s_r$ retaining their meaning on Borel sets on such curves.

However, the following remarks show that if there appear points for
which $\theta = 0$ there may occur essential singularities.

2) If at the endpoint $X_0$ of the curve $\mathcal{S} = X_t (0 \leq t \leq 1)$ the complete angle $\theta = 0$,
then Theorem 4 generally speaking ceases
to be valid. This may be observed in the
example of a geodesic curve on a spire-
shaped surface as depicted in Figure 110.

3) Such a curve $L$ (see Figure 110) in
the large may fail to be rectifiable (for
the construction of an example see p.78) i.e., in an arbitrarily small
neighborhood of the point $X_0$ its length will be infinite. Analogously
one may construct a nonrectifiable simple arc with rotation of bounded
variation, if on it (inside it) there lies a point $X_0$ with $\theta = 0$ (Figure 111).

Already using Figure 101 it has been shown that, in a bounded region
with curvature $\omega^+ \geq 2\pi$, there can lie arbitrarily long curves with rotations
of uniformly bounded variations. Figures 110 and 111 show that this can
happen even for sequences of polygonal curves. We are thinking of poly-
gons approximating from one side the curves $\mathcal{S}$ depicted there.

4) In defining the concept of rotation, we have used polygons lying on
one side of the arc in question. The question arises: if the simple arc $\mathcal{S}$
has a rotation of bounded variation in the sense of Definition 1, then do
polygons inscribed in it, provided they are sufficiently fine in the sense
of subdivision with respect to the parameter $t$ on $\mathcal{S}$, necessarily have
rotations of uniformly bounded variations?

One may see from Figure 110 that this certainly may fail to be the case, if there is on $\mathcal{L}$ a point with angle $\theta = 0$. One need only take a polygon in Figure 110 starting with $A_1A_2$ and then follow along $\mathcal{L}$ to $A_3$, then $A_3A_4$, then again along $\mathcal{L}$ and so forth. Is this kind of singularity obviously connected with the presence on $\mathcal{L}$ of points where $\theta = 0$?

5) In the case of intrinsic geometry of convex surfaces, points where $\theta = 0$ do not exist and all the singularities mentioned in this subsection are missing.

7. Rectifiability of curves with rotation of bounded variation not passing through singular points.

**Lemma 7.** Suppose that $O$ is a fixed nonsingular point and that $U(r)$ is a convex neighborhood of the point $O$ homeomorphic to the disc, all of whose points are distant from $O$ by no more than $r$. Then the least upper bound $L$ of the lengths of simple curves $\mathcal{L}$ situated in $U(r)$ and having rotations $\sigma_r(\mathcal{L}) \leq M$ of uniformly bounded variation tends to zero along with $r \to 0$.

**Proof.** Case 1. Suppose that $\omega(O) = \pi - 3\delta < \pi$. We take $r$ so small that already all the points distant from $O$ by a distance not greater than $2r$ lie in a neighborhood $V$ homeomorphic to the disc for which

$$\omega^+(V - O) < \delta.$$  

We consider an arc $\mathcal{L}$ lying in $U(r)$ with variation of rotation $\sigma_r(\mathcal{L}) \leq M$. The arc $\mathcal{L}$ may be decomposed into $2M/\delta$ pieces such that the variation of the rotation on each of them does not exceed $\delta$. Applying Lemma 6 to each of the pieces, we verify that $\mathcal{L}$ is rectifiable and that its length satisfies the inequality

$$\mathcal{L} < \frac{4Mr}{\delta \cos (\pi - \delta/2)},$$

from which it follows that $\mathcal{L} \to 0$ as $r \to 0$.

Case 2. Suppose that $\omega(O) \geq \pi$. By hypothesis, the complete angle $\theta > 0$ at $O$. We denote by $n \geq 2$ the integer part of the ratio $2\pi/\theta$. Cut the region $U(r)$ along one of its radii and paste together $n$ of its exemplars, identifying the vertices $O$ and successively pasting the edge radii. In the newly obtained metric space $\omega(O) = 2\pi - n\theta = \pi - 3\delta < \pi$.

If the radius $r$ is so small that in the original metric

$$\omega^+(V - O) < \frac{\delta}{n},$$
then consider the curve $\mathcal{L}$ transferred to the newly pasted metric space. We again have inequality (19), from which it follows that Lemma 7 is valid.

**Theorem 5.** If the simple arc $\mathcal{L}$ is the limit of the simple arcs $\mathcal{L}_n$, whose rotation is of uniformly bounded variation, $\sigma(\mathcal{L}_n) \leq M$, and if there are no singular points on $\mathcal{L}$, then all the $\mathcal{L}_n$ are rectifiable, their lengths uniformly bounded, the arc $\mathcal{L}$ is rectifiable and the lengths of the $\mathcal{L}_n$ converge to the length of $\mathcal{L}$.

**Proof.** Taken an arbitrarily small $\varepsilon > 0$, and then a small number $0 < \delta < \pi/3$. On the curve $\mathcal{L}$ there are only finitely many points with curvature exceeding $\delta$, say $m_1$ such points. From Lemma 7, there exists an $r > 0$ so small that the length of the curves $\mathcal{L}_n$ lying entirely in a neighborhood $U(r)$ of one of these points is less than $\varepsilon/m_1$.

Around each point $X \in \mathcal{L}$ we choose a neighborhood $U$ of radius less than $r$ and such that even a neighborhood of a radius twice as large is contained in a neighborhood $V$ homeomorphic to the disc and with curvature $\omega^+(V - X) < \delta$. Each region $U$ covers on $\mathcal{L}$ some open interval around a point $X$. From these intervals, according to the Borel lemma, one may select a finite covering of the arc $\mathcal{L}$ formed by regions $U_i$ ($i = 1, 2, \cdots, m_1 + m_2$) with centers $X_i$ enumerated in the order of succession on $\mathcal{L}$. Among these points there are a fortiori $m_1$ of the earlier mentioned points.

Correspondingly $\mathcal{L}$ may be divided into $m_1 + m_2$ pieces so that each of them along with some $\rho$-neighborhood is covered by one of the regions $U_i$.

We choose a curve $\mathcal{L}_n$ with an index so large that each of its points is distant by less than $\rho > 0$ from the corresponding (with respect to the parameter on the curve) point of $\mathcal{L}$. The curve $\mathcal{L}_n$ may be divided into $m_0 \leq 2M/\delta$ pieces with the variation of the rotation less than $\delta$. Supplementing with appropriate points of division, we divide it into $m_0 + m_1 + m_2$ pieces lying in the corresponding regions $U_i$ and having rotations less than $\delta$.

Pieces falling in the neighborhood of the distinguished $m_1$ points, because of the choice of $r$, are small in total length, and the lengths of the remaining pieces, from Lemma 6, do not exceed

$$\frac{2r}{\cos(3\delta/2)} \left( m_0 + m_1 + m_2 \right).$$

Thus all the $\mathcal{L}_n$ are rectifiable, their lengths bounded uniformly by
some number $N$, and the limit curve $\mathcal{L}$ also rectifiable.

In order to verify that the lengths of the $\mathcal{L}_n$ converge to that of $\mathcal{L}$, we repeat our construction with the following additional requirements:

1) we suppose that $\delta$ was chosen so small that

$$1 - \frac{1}{\cos((3/2)\delta)} < \frac{\varepsilon}{N};$$

2) suppose that $m_3$ is the number of links of a polygon inscribed in $\mathcal{L}$ with a length different from that of $\mathcal{L}$ by less than $\varepsilon$. We suppose that the quantity $\rho$ chosen earlier satisfied one more requirement:

$$\rho < \frac{\varepsilon}{2(m_0 + m_1 + m_2 + m_3 + 1)}.$$

We assert that then if $\mathcal{L}_n$ and $\mathcal{L}$ are closer than $\rho$ we have the following inequality for the lengths:

(20) $|\mathcal{L} - \mathcal{L}_n| < 4\varepsilon$.

Indeed, suppose that $\mathcal{L}_n$ is subdivided as indicated above into $m_0 + m_1 + m_2 + m_3$ pieces. Those which fell into the $m_1$ particular regions $U_i$ do not, because of the choice of $r$, exceed $\varepsilon$ in total length. The lengths of the limit curve $\mathcal{L}$ in these regions also do not exceed $\varepsilon$. Thus we cannot here accumulate a difference in length larger than $\varepsilon$. On the remaining pieces, by condition (1) and Lemma 6, the total length of $\mathcal{L}_n$ differs from the sum of the lengths of the shortest arcs spanning the endpoints of these pieces by less than $\varepsilon$. In view of condition (2) these shortest arcs altogether differ less than $\varepsilon$ from the length of the corresponding links of the polygon inscribed in $\mathcal{L}$. This last, by the choice of the pieces $m_3$, is in its totality close in length to the remaining piece of $\mathcal{L}$. Therefore we obtain inequality (20) and along with it the convergence of the lengths.

Theorem 5 is proved.

**Corollary 1.** Every simple arc $\mathcal{L}$ with rotation of bounded variation and not having on the interior or at the endpoints singular points with $\theta = 0$ is rectifiable. Simple polygons converging to it on one side and having rotations of uniformly bounded variation, as constructed in Theorem 3, converge to $\mathcal{L}$ along with the lengths.

**Corollary 2.** Every simple arc $\mathcal{L}$ with rotation of bounded variation admits one-sided approximation by simple arcs, whose lengths converge to the length of $\mathcal{L}$.
3. Second theorem on pasting.

8. Pasting of a manifold from pieces bounded by curves with rotation of bounded variation. Suppose that \( G_i \ (i=1, \ldots, m) \) are compact regions with edge, distinguished from certain manifolds \( R_i \) of bounded curvature and having a boundary in the form of a finite number of simple closed curves. Each curve is supposed to be subdivided into a finite number of curvilinear links. All the links are supposed to be curves with rotations of bounded variation in the original spaces \( R_i \). The links and their endpoints, by hypothesis, do not contain singular points with \( \theta = 0 \), so that all the links are certainly rectifiable.

Suppose further that the distinguished regions \( G_i \) are pasted, along each link there being pasted two links of equal length; moreover along pieces which correspond in length. Around the vertices the pasting is supposed to be realized successively, as with the sectors of one plane disc. The set of all points of the already pasted space is converted into a topological space in the usual way for pasting.

We shall suppose that no free links remain under the pasting, so that the pasted space is a closed two-dimensional manifold. This last requirement could be dropped, admitting that some pieces of the boundary remain along which the pasting is not carried out. But for such a manifold with edge the boundary will consist of a finite number of simple closed curves. Pasting along each of them a sufficiently high cylinder with the outline of its base corresponding in length, we obtain a closed manifold, in which the original manifolds with edge being pasted become convex regions. Therefore we restrict ourselves to the case of pasting of a closed manifold.

In the pasted manifold \( R \) we distinguish a class \( K \) of curves to which we assign all curves consisting of a finite number of simple arcs, each of which except possibly for its endpoints lies inside one of the pieces \( G_i \). The endpoints of the distinguished arcs may lie on curves and at vertices of the pasting. To a curve \( \mathcal{L} \in K \) we may assign a finite or infinite length, measuring it along the pieces in the corresponding spaces \( R_i \). We introduce in \( R \) the function

\[
(21) \quad \rho(X, Y) = \inf_{\mathcal{L} \in K} s [\mathcal{L}(X, Y)],
\]

where the infimum is taken with respect to all curves \( \mathcal{L} \in K \) joining the points \( X \) and \( Y \).

Lemma 8. The value of \( \rho(X,Y) \) does not decrease if the class \( K \) is ex-
tended by admitting also curves \( \mathcal{L} \) with a finite number of simple pieces entirely passing along curves of the pasting.

This follows immediately from Corollary 2 of Theorem 5.

**Lemma 9. For any pair of points \( X, Y \), the value of \( \rho(X, Y) \) is finite.**

This follows from Lemma 1 and the connectedness of the net of curves of the pasting.

Moreover, the two following assertions are true.

**Lemma 10. The function \( \rho(X, Y) \) is a metric.**

**Lemma 11. The metric \( \rho \) defines in the pasted manifold the same topology as was defined by the process of pasting itself.**

Now we may formulate the fundamental assertion.

**Theorem 6 (Second theorem on pasting).** The manifold \( \mathbb{R}^2 \) with metric \( \rho(X, Y) \), pasted according to the rules described above, and with \( \rho \) given by (21), is a two-dimensional manifold of bounded curvature.

**Corollary.** In a sufficiently small neighborhood of any point which is interior for one of the pasted regions \( G_i \), the metric \( \rho \) coincides with the original metric \( \rho_i \) of the space \( \mathbb{R}^2 \). Therefore within the region \( G_i \) the curvatures of sets are preserved. In the pasted manifold, by the same principle, one-sided rotations of curves of the pasting are preserved, and also, at vertices the adjacent sector angles \( \theta_i \). The curvatures \( \omega_1, \omega_2 \) of the curves of the pasting and of the vertices of the pasting are defined by the indicated rotations and angles according to the usual rule

\[
\omega_1 = \tau_r + \tau_i, \quad \omega_2 = 2\pi - \sum \theta_i.
\]

9. **Plan of proof of Theorem 6.** In order to establish Theorem 6, we may attempt to verify directly that the axioms of boundedness of curvature are satisfied in \( \mathbb{R}^2 \) by a method similar to that used in the proof of the first theorem on pasting, Theorem 7 of Chapter VI. But it will be easier to show that the metric \( \rho \) may be obtained as the limit of some uniformly converging sequence of metrics \( \rho_n \) whose curvatures are uniformly bounded. From the results of Chapter IV it follows that Theorem 6 will then be valid. We shall hold our exposition of the proof to such a plan.

**Approximating metrics.** In order to construct the approximating metric \( \rho_n \), we carry out in each of the \( \mathbb{R}^n \) where \( \mathcal{G}_i \) lies, the following construction.

1. Each vertex of \( \mathcal{G}_i \) is encircled in a convex polygon \( Q \) of small \( \langle 1/n^3 \rangle \) radius and small perimeter. We suppose further that \( Q \) lies in a
neighborhood with small curvature (aside from the vertex itself).

2. Each link of the boundary \( \mathcal{G}_i \) is approximated from within \( \mathcal{G}_i \) in such a way that the variation of its rotation be little \( (< 1/n^3) \) different from the variation of the rotation of the link itself, that the points of these curves are distant by less than \( 1/n^3 \) from the points on the approximated piece of the side (link of the boundary) of \( \mathcal{G}_i \), which correspond to them with respect to the parameter (relative length), that the length of the polygon differ by little \( (< 1/n^3) \) from the length of the piece of the boundary being approximated, that the endpoints of each polygon lie on the approximated links at the same distance \( \varepsilon_n > 0 \) from its endpoints, and finally that the resulting polygons do not touch one another and do not form superfluous intersections with the polygons \( \mathcal{Q} \). All of this is schematically depicted in Figure 112. ³

3. Now the region \( \mathcal{G}_i \) decomposes into regions of three types: a polygonal interior region \( \mathcal{G}^i \) heavily outlined in Figure 112, narrow regions \( \mathcal{G}^i'' \) adjacent to the sides of \( \mathcal{G}^i \), and regions \( \mathcal{G}^i''' \) adjacent to the vertices of \( \mathcal{G}^i \). The last are cross-hatched in Figure 112.

4. We select only the regions \( \mathcal{G}^i \). If earlier along a link of length \( l + 2\varepsilon_n \) the region \( \mathcal{G}_i \) was pasted with another region, then for \( \mathcal{G}_i \) the corresponding piece of the boundary does not have entirely the same length \( l' \). We paste to \( \mathcal{G}^i \) along \( l' \) a planar trapezoidal strip of width \( 1/n \) with bases

³ We suppose the \( \varepsilon_n \) so small that the endpoints of the polygons lie within the polygons \( \mathcal{Q} \).
SECOND THEOREM ON PASTING

Only after this we paste the regions $\bar{G}_i$, along the sides, in a manner analogous to the pasting of the regions $\bar{G}_i$ in $R$. Here we obtain along each "side" between the regions $\bar{G}_i'$ so to speak a "building" in the form of a pair of narrow, nearly right-angled trapezoids (Figure 113).

5. In the neighborhood of the vertices of $\bar{G}_i$ there remain "holes". Into each of them we paste a right circular cylinder with a base of the appropriate perimeter and a height equal to half that perimeter.

6. According to the first theorem on pasting the resulting space $R_n$ will have an intrinsic metric $\rho_n$ of bounded curvature. It is easy to verify that the absolute curvatures of these metrics are uniformly bounded.

Transfer of the metrics $\rho_n$ to a single region of representation. We compare (topologically) the region $R_n$ in which $\rho_n$ is given with the basic manifold $R$. To this end we repeat in $R$ all the subdivisions of $\bar{G}_i$ into parts $\bar{G}_i'$, $\bar{G}_i''$, $\bar{G}_i'''$. We map each piece $\bar{G}_i''$ onto the trapezoidal strip replacing it and map onto one of the bases of the trapezoid the curve of division of $\bar{G}_i'''$ and $\bar{G}_i''$, with exact correspondence in length, and we map the other base of the trapezoid onto the piece of the region $\bar{G}_i$ adjacent to $\bar{G}_i''$.

Finally, we map the pieces of the pasted cylinders cross-hatched in Figure 114 topologically onto the regions $\bar{G}_i'''$, making the mapping consistent with the mapping of the pieces of the boundary already established.

Further we shall show what exactly is the correspondence of the trapezoidal regions and the regions $\bar{G}_i''$. On the side of $\bar{G}_i$ we lay off points dividing the side into pieces of length of order $1/n^2$. From each such point $A$ we draw in $R_i$ a shortest arc $AM$ up to the polygon $l$ ap-
proximating the side, as in Figure 115. Some piece BM of this curve divides $G''$. Inasmuch as the "width" of $G''$ is of order $1/n^3$, and the distance of the points $A$ of order $1/n^3$, these divisions occur independently of one another. The entire region $G''$ is subdivided into "cells". Correspondingly (in terms of the parameters along the bases) we divide up the plane strip and topologically match $G''$ and the strip along cells, adjusting only the relations on the junctions of the cells and the edges of the region and the strip.

After all these relations have been established, the metrics $\rho_n$ may be considered to be given on $R$.

Estimate of $\rho$ from below. Suppose that $X$ and $Y$ are two fixed points in $R$, not lying on the curves of the pasting. For any $\varepsilon > 0$, from the definition of $\rho(X,Y)$, there exists a curve $\mathcal{L} \subseteq K$ joining $X$ and $Y$ with the length of $\mathcal{L}$ satisfying

$$\rho(X, G) > \mathcal{L} - \varepsilon.$$  

Figure 116.  

For sufficiently large $n$ the endpoints and basic portions (with respect to length) of the segments of $\mathcal{L}$ will lie inside the regions $\tilde{G}_i''$, as in Figure 116. These pieces, directly in the structure of $\tilde{G}_i''$, carry over into $R_n$, as in Figure 117. There they may be augmented to a continuous curve $\mathcal{L}'$ with a small additional expenditure of length, i.e., for $n \geq N(\varepsilon)$

$$\rho > \mathcal{L} - \varepsilon > \mathcal{L}' - 2\varepsilon \geq \rho_n - 2\varepsilon,$$

so that

$$\rho(X, Y) \geq \limsup_{n \to \infty} \rho_n(X, Y).$$  

(22)
Estimate of \( \rho \) from above. Write \( A = \lim \inf_{n \to \infty} \rho_n(X,Y) \). It follows from (22) that \( A \) is finite. In what follows we shall preserve only those \( n \) for which \( \rho_n(X,Y) \to A \).

Suppose that \( n \) is so large that \( X \) and \( Y \) lie inside the regions \( G'_i \) and \( \rho_n(X,Y) < A + 1 \). Join \( X \) to \( Y \) in \( \mathbb{R}^n \) by a shortest arc \( L_n \). This shortest arc certainly does not touch the cylinders pasted up in the construction. If it touches a pair of strips pasted up in the construction, then it already certainly intersects them. The number of such passages through the strips is not greater than \((A + 1)n/2\), since on each intersection the length decreases by at least \( 2/n \).

We construct a curve \( \mathcal{L} \) joining \( X \) and \( Y \) in \( \mathbb{R} \) as follows. First we transfer to \( \mathbb{R} \) those pieces of \( \mathcal{L}_n \) which lay in the regions \( \overline{G}'_i \). These pieces still do not form a continuous curve, since they are broken off on passing through the regions \( \overline{G}''_j \). To each intersection \( MN \) of the shortest arc with \( \mathcal{L}_n \) a plane strip we assign a piece \( M'N' \) on the curve of the pasting of the pair of strips in \( \mathbb{R}_n \) (Figure 118) and the piece \( M''N'' \) corresponding to it on the curve of the pasting in \( \mathbb{R} \). We note that in length \( M''N'' = M'N' \leq MN \). Finally, we join \( MM'' \) and \( N''N \) by sufficiently short curves in \( \mathbb{R} \).

![Figure 118.](image)

The supplementary pieces \( MM'' \) and \( N''N \) will have lengths of order \( 1/n^2 \), while their total number does not exceed \((A + 1)n \). Therefore for large \( n \) these pieces will be sufficiently small in total, and we may verify that

\[
\mathcal{L} \leq \rho_n + \varepsilon,
\]

from which follows the validity of the inequality \( \rho(X,Y) \leq A \).
Uniformity of the convergence \( \rho_n \to \rho \). From (21) and (22) it follows that for each fixed pair of points \( X, Y \in R \), not lying on the curves of the pasting, \( \rho_n(X, Y) \to \rho(X, Y) \). It remains to prove that this convergence holds (uniformly!) for any pair of points \( X, Y \in R \).

We formulate an auxiliary assertion.

**Lemma 12.** For any \( \varepsilon \geq 0 \) there exists an \( N \) and a finite system of points \( A_j \), not lying on the curves of the pasting, such that for \( n \geq N \) all the \( A_j \) lie inside the regions \( G_i \) and have the property that any point \( X \in R \) may be put into correspondence with one of the points \( A_j \), this being done in such a way that for all \( n \geq N \) we will have the inequalities

\[
\rho(X, A_j) < \varepsilon, \quad \rho_n(X, A_j) < \varepsilon.
\]

Uniform convergence of \( \rho_n \) to \( \rho \) on the entire manifold \( R \) follows from Lemma 12 because of the following inequalities:

\[
| \rho_n(X, Y) - \rho(X, Y) | \leq \rho_n(X, A) + \rho(X, A) + \rho_n(Y, B)
+ \rho(Y, B) + | \rho_n(A, B) - \rho(A, B) | < 5\varepsilon,
\]

where \( A \) and \( B \) are points of the system \( A_j \) corresponding to the points \( X \) and \( Y \). The first four terms of the right side are small by the construction of the net of points \( A_j \), and the last term is small for large \( n \) since \( \rho_n \to \rho \) for each pair of fixed points \( A_j = A, A_k = B \).

**Plan of the proof of the lemma.** We shall outline the construction of the net of points \( A_j \) and the assignment of points \( X \in R \) to points of this net.

1. For some \( \delta > 0 \) we choose in the regions \( G_i \) in the original spaces \( R_i \) some \( \delta \)-net. We preserve from it only those points which lie strictly inside \( \overline{G}_i \). This will be the “first net”. On each side of each region \( \overline{G}_i \) we choose points \( C \) forming a \( \delta \)-net in the sense of pieces along the length of the side. We shift these inside \( G_i \) by less than \( \delta \). This will be the “second net”. Together, for sufficiently small \( \delta \), they constitute the net of the \( A_j \).

2. We turn to the “attachment” of the points \( X \in R \) to the points \( A_j \). First we consider all points \( X \in G_i \) distant by more than \( 3\delta \) from the boundary of \( \overline{G}_i \). Each such point may be joined in \( R_i \) by a shortest arc to the closest point \( A \) of the first net. We attach \( X \) to this point. The entire shortest arc \(XA\) lies inside \( G_i \) at a distance of at least \( 2\delta \) from the boundary of \( G_i \).

To each point \( X \) of the \( 3\delta \)-edge of the region \( G_i \) we join some point
of the second net by a curve consisting of the following pieces: a) a shortest arc $XB$ from $X$ to the boundary of $\overline{G}_i$; b) a path $BC$ along the boundary to the closest point $C$ (a still unshifted point of the second net) in a direction fixed in advance; c) a path $CA$ along which the point $C$ was shifted inside $\overline{G}_i$. The length of the entire path satisfies $XB + BC + CA \leq 3\delta + \delta + \delta$. We attach each point $X$ to the point $A$ obtained along such a path.

If $5\delta < \epsilon$, the first of the requirements (23) is satisfied.

3. Now we choose $n$ so large that the following conditions are satisfied: a) all the points $A_j$ chosen above lie inside the regions $G'_i$; b) each region $G'_i$ contains all the points of the region $G_i$ distant by more than $\delta$ from the boundary of $G_i$; c) the boundaries of the $G'_i$ will be close to the boundaries of the $G_i$ in the sense of $\delta$-closeness of the points corresponding with respect to the parameter, with respect to relative length; d) the mappings of the regions $G'_i$ onto plane strips and the regions $G'''_i$ onto cylinders preserves the $\delta$-closeness of points of the regions to the boundary points with similar values of the parameter on the boundary.

4. It remains to observe that for sufficiently small $\delta$ and sufficiently large $n$ the second of the inequalities (23) is satisfied.

For points $X \in G_i$ distant by more than $3\delta$ from the boundary of $G_i$, this is certainly satisfied, since the entire path $XA$ lies in $G'_i$ and therefore preserves its length in the metric $\rho_n$.

Consider a point $X$ in the $3\delta$-edge of the region $G_i$. In the metric $\rho$ the point $X$ is joined to $A$ by the path $XBACA$, as in Figure 119. In the metric $\rho_n$, there remains unchanged some initial piece $XB'_i$ of the shortest arc $XB$. The point $B'_i$ on the boundary of $\overline{G}'_i$ corresponds, in the parameter along the boundary, to some point $B''_i$ on the boundary of $\bar{G}_i$. Analogously the curve $CA$ has a piece $C'_iA$ in $\overline{G}'_i$. In the metric $\rho_n$ the point $X$ may be joined to $A$ by the path $XB'_iB''_iBCC''_iC'_iA$.

For sufficiently small $\delta$ and large $n$ this path will be smaller than $\varepsilon$.

The cases $X \in G'''_i$ or $X \in G'''_i$ introduce only nonessential changes in the above construction.

In such a way the validity of Lemma 12 is established, and along with it the uniform convergence of metrics $\rho_n \to \rho$, and from this Theorem 6.

10. Lemmas on sequences of polygons.

Lemma 13. Suppose that the endpoints $Y_n$ of the simple polygons $L_n$ converge to a point $O$ with a complete angle $\theta \equiv 0$, and the origins $X_n$ of these polygons remain all the time outside some fixed neighborhood of the point $O$. Then for sufficiently large $n$ one may in an arbitrarily small neighborhood of the point $O$ replace the terminal portion of each polygon $L_n$ by a shortest arc going into $O$, while the polygons remain simple and the variation of their rotations does not increase by more than some $\varepsilon > 0$ given in advance.

Proof. Consider the neighborhood $U_r$ of the point $O$ where $r < \varepsilon$ and $U_r$ lies along with the shortest arcs joining its points in an absolutely convex neighborhood of the origin $V$ with absolute curvature $\Omega(V - O) = u$ so small that

$$u < \frac{\varepsilon}{3}, \quad u < \theta, \quad \left(\frac{3}{2} \pi + \theta\right) \frac{u}{\theta - u} < \frac{\pi}{4}$$

and all the $X_n$ lie outside $V$.

We choose a polygon $L_n$ with index so high that

$$\rho(O, Y_n) < r e^{-\frac{2\pi}{\tan(\varepsilon / 2)}}.$$  

We take $Y_n$ to be the closest point to $O$ on $L_n$ (otherwise the endpoints may be dropped).

Suppose that $M$ is the first and $N$ the last of the points of $L_n$ distant from $O$ by exactly $r$ ($M$ and $N$ may be the same). If the variation of the rotation $\sigma(NY_n) > (\pi / 2) - \varepsilon$, then we replace $MY_n$ by the shortest arc $MO$. From the choice of $MO$ it follows that $MO$ forms with the preceding part of the polygon on both sides angles not less than $\pi / 2$, from which it follows that the variation of the polygon increases if at all by less than $\varepsilon$.

Now suppose that

$$\sigma(NY_n) \leq \frac{\pi}{2} - \varepsilon.$$  

We join the point $O$ with all the vertices $N = A_0, A_1, \cdots, A_m = Y_n$ of the polygon $NY_n$. This may be done in such a way that one obtains reduced triangles $OA_0A_1, OA_1A_2, \cdots, OA_{m-1}A_m$.

Case 1. Suppose that the sectors of these triangles adjacent to $O$ follow
in a strictly monotone order around the point $O$. Then we develop these triangles on the plane and distribute them in the same sequence around the point $O'$. Suppose that $\phi_i$ are the angles of the sectors adjacent to $O$ ($i = 1, \ldots, m$), $\phi'_i$ the corresponding angles of the plane triangles, $\alpha_j$ the sector angles of the triangles at the vertices $A_j$ ($j = 1, \ldots, 2m$), $\omega_i$ the curvatures of the triangles including the interior rotations of their sides.

We shall prove that $\sum \phi'_i < 2\pi$.

We have:

$$\sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} \phi_i + \sum_{j=1}^{2m} \alpha_j - \pi m. \tag{27}$$

Moreover

$$\tau(A_2 + \cdots + A_{m-1}) = (m - 2)\pi - \sum_{j=2}^{2m-1} \alpha_j \leq \sigma(NY_n) < \frac{\pi}{2}. \tag{28}$$

From (27) and (28) it follows that

$$\sum_{i=1}^{m} \phi_i \leq \sum_{i=1}^{m} \omega_i + \pi m - \sum_{j=2}^{2m-1} \alpha_j < ku + \frac{3}{2}\pi$$

where $k$ is the largest number of coverings of one point by the triangles in question. But evidently $\sum \phi_i \geq (k-1)\theta$, so that

$$(k-1)\theta \leq \frac{3}{2}\pi + ku,$$

from which

$$ku \leq \left(\frac{3}{2}\pi + \theta\right)\frac{u}{\theta - u} < \frac{\pi}{4}.$$

Finally,

$$\sum \phi'_i \leq \sum \phi_i + ku \leq 2ku + \frac{3}{2}\pi < 2\pi.$$

Now we show that at least one ray $O'A'_i$ forms with the segment $A_iA'_{i+1}$ an angle less than $\varepsilon/2$. In fact otherwise it would be possible to pass through the point $N$ a logarithmic spiral

$$\rho = re^{-\phi \tan(\varepsilon/2)}$$

with center $O'$, as in Figure 120. This spiral forms with the radii the angle $\varepsilon/2$ and will pass closer to $O'$ than our polygon. But this spiral, even after a complete revolution, approaches $O'$ only at the distance

$$re^{-\frac{2\pi}{\tan(\varepsilon/2)}}.$$
which contradicts condition (25) on the smallness of $O'Y'$. 

Thus, for some point $\angle O'A_iA'_{i+1} < \varepsilon/2$. Replacing $A_iY_n$ by the shortest arc $AO$, we obtain the required change in the polygon.

**Case 2.** It remains to consider the case when the order of succession of the sectors of the triangles adjacent to $O$ is violated. Suppose it is first violated by the triangle $OA_iA_{i+1}$.

![Figure 120.](image1)

![Figure 121.](image2)

We first suppose that the link $A_iA_{i+1}$ goes outside the sector of the preceding triangle (Figure 121). Then we consider the piece from $A_i$ to the following intersection with $OA_i$ or to the endpoint $Y_n$. We mark off on it the point $T$ closest to $O$ and consider the rotation of the contour $OA iTO$ enclosing the region $g$.

From the choice of $T$ and the fundamental properties of angles and rotations, we have in the notation indicated in Figure 121:

$$\beta \geq \frac{\pi}{2}, \quad \phi \geq 0, \quad \gamma \leq \pi + \omega^+(OA_iA_{i+1}),$$

$$\alpha = 2\pi - \gamma - \omega(A_i), \quad \tau(OT) \leq 0, \quad \tau(OA_i) \leq 0,$$

$$\omega(g) + \pi - \phi + \pi - \alpha + \pi - \beta + \tau(OT) + \tau(OA_i) + \tau(A_iT) = 2\pi,$$

which gives

$$\tau(A_iT) \geq \frac{\pi}{2} - \omega^+(OA_iA_{i+1}) - \omega(A_i) - \omega(g) > \frac{\pi}{2} - \varepsilon,$$

in contradiction to inequality (26).

This means that $A_iA_{i+1}$ passes through the sector $A_i$ of the triangle
Converging Curves

As in Figure 122. In this case we replace \( A_iY_n \) by \( OA_i \). We have

\[
\gamma \leq \pi + \omega^+(OA_{i-1}A_i),
\]

\[
\pi - \alpha = \omega(A_i) + \gamma - \pi \leq \omega(A_i) + \omega^+(OA_{i-1}A_i).
\]

Now we compare the rotations \( \tau^0 \) and \( \tau^1 \) of the original and changed polygons at the vertex \( A_i \). For \( \alpha \geq \pi \) we have

\[
|\tau^0_i(A_i)| = |\alpha + \beta - \pi - \alpha - \pi = |\tau^1_i(A_i)|,
\]

and for \( \alpha < \pi \)

\[
|\tau^1_i(A_i)| = \pi - \alpha \leq \omega(A_i) + \omega^+(OA_{i-1}A_i) < \frac{\varepsilon}{3},
\]

i.e., the rotation of the new polygon at the vertex \( A_i \) is less than the previous rotation, or is in general small. We note further, that for the shortest arc \( OA_i \) the variation of the rotation is certainly less than \( \varepsilon/3 \).

Lemma 13 is proved.

**Lemma 14.** If the simple arc \( \mathcal{Q} \) is the limit of simple polygons \( L_n \) which have rotations of uniformly bounded right rotation, i.e.

\[
\sigma_r(L_n) \leq S,
\]

then \( \mathcal{Q} \) and each of its points has a definite direction to right and left.

It suffices to carry out the proof for the endpoint \( A \) of the arc \( \mathcal{Q} \). If the complete angle \( \theta \) at \( A \) is equal to zero, then \( \mathcal{Q} \) has a direction as does any curve issuing from \( A \). We shall suppose that \( \theta(A) \neq 0 \). Then the proof of Lemma 14 simply follows the proof of Theorem 2. If in formula (10) of the present chapter, with the curve \( L_n \) in the place of \( \mathcal{Q} \), one replaces \( \sigma \) by \( S \), then formula (10) remains true. In the proof of Theorem 2 the situation depicted for a piece of \( \mathcal{Q} \) in Figure 105, led to a contradiction. This time the same situation for a piece of \( L_n \) leads to the conclusion that on all \( L_n \) for sufficiently large \( n \) in the vicinity of the piece \( EB \) there exists a charge \( \sigma \), comparable with \( \alpha \) (see Figure 105). Since this may be repeated on pieces ever closer to \( A \), for sufficiently large \( n \) there are so many pieces with \( \sigma, \approx \alpha \) that this leads to a contradiction with inequality (29).

This proves the validity of Lemma 14.

**Lemma 15.** Suppose that a sequence of simple polygons \( L_n \) with endpoints
$A_n, B_n$ converges to a simple arc $\mathcal{L}$, the variation of the right rotations of these broken curves being bounded uniformly:

\[(30) \quad \sigma_r(L_n) \leq S.\]

Suppose further that the complete angle $\theta \equiv 0$ at some point $C$ on $\mathcal{L}$. Then beginning with sufficiently large $n$ one may replace $L_n$ by a polygon $L'_n$, by rejecting some piece $M_nN_n$ of $L_n$ and replacing it by a pair of shortest arcs $M_nC$ and $CN_n$. In addition one may adjoin the following conditions:

1) the variation takes place in an arbitrarily small neighborhood of the point $C$ given in advance;

2) the shortest arcs $CM_n$ and $CN_n$ have no common points other than $C$ or an initial segment adjacent to $C$;

3) the pieces $A_nM_nC$ and $CN_nB_n$ of the altered polygon $L'_n$ are each separately simple arcs;

4) the right rotations of $L'_n$ at the points $M_n$ and $N_n$ are less in absolute value than some $\varepsilon > 0$ given in advance;

5) the shortest arcs $M_nC$ and $CN_n$ form at the point $C$ angles less than $\varepsilon$ with the branches of $\mathcal{L}$;

6) if $C$ is not a cusp on the limit curve $\mathcal{L}$, (i.e., the branches of $\mathcal{L}$ do not form a zero angle at it), then $L'_n$ remains a simple polygon in the large;

7) if $C$ is a cusp, then the shortest arc $M_nC$ essentially intersects the piece $N_nB_n$ not more than a finite number of times, and similarly for the shortest arc $CN_n$ and the piece $A_nM_n$;

8) after altering $L_n$ to $L'_n$ the variation of the right rotation of $L'_n$ on the open portions $A_nC$ and $CB_n$ and the absolute value of the right rotation at the point $C$ on the limit curve $\mathcal{L}$ do not in sum significantly exceed $\sigma_r(L_n)$, or more precisely

\[(31) \quad \sigma_r'(A_nC) + \sigma_r(C) + \sigma_r'(CB_n) < \sigma_r(L_n) + 2\Omega(C) + \varepsilon,\]

where the primes denote the variations of the rotations on the altered polygons.

**Proof.** 1. Suppose that $C_n$ are points on $L_n$ corresponding in parameter on the converging curves $L_n \to \mathcal{L}$ to the point $C$ on $\mathcal{L}$. In accordance with Lemma 13 we will replace certain endpieces $M'_nC_n$ of the pieces $A_nC_n$ by shortest arcs $M'_nC$. We carry out this construction in ever smaller neighborhoods of the point $C$, i.e., such that $M'_n \to C$ as $n \to \infty$. Moreover, we ensure that $A_nM'_nC$ remain simple arcs and that the variations of the right rotations increase by less than $\varepsilon/2$. If for an infinite number of altered polygons there is formed an absolute rotation at the point $M'_n$.
which exceeds \( \varepsilon \), i.e., if condition 4 of Lemma 15 under proof is not satisfied, this means that on the substituted pieces \( \sigma_r(M_n^1C_n) \geq \varepsilon/2 \). Then we reject the pieces \( M_n^1C_n \) and apply Lemma 13 to the shortened polygons \( A_nM_n^1 \), replacing the endpieces \( M_n^2M_n^1 \) by the shortest arcs \( M_n^2C \). If again for an infinite number of altered polygons \( \sigma_r(M_n^1) \geq \varepsilon \), then for them \( \sigma_r(M_n^2M_n^1) \geq \varepsilon/2 \). Repeating this contraction more than \( 2S \varepsilon^{-1} \) times, we arrive at a contradiction with the estimate (30). This means that condition 4 of Lemma 15 may be satisfied.

2. Condition 1 of Lemma 15 may be considered as satisfied, since the construction may be carried out in a neighborhood of the point \( C \) agreed on in advance. Moreover, the shortest arcs \( M_nC \) and \( CN_n \) may each time be drawn so that conditions 2 and 7 of Lemma 15 are satisfied.

3. From Lemma 14, curve \( \mathcal{A} \) has at each point, including \( C \), definite directions of its branches. We select on \( \mathcal{A} \) a point \( X_0 \) such that the piece \( X_0C \) of the curve \( \mathcal{A} \) lies in an absolutely convex neighborhood \( U \) with an absolute curvature \( \Omega(U - C) < \delta \) and such that for all \( X \in X_0C \) the shortest arcs \( XC \) form with \( \mathcal{A} \) at the point \( C \) angles less than \( \delta \).

Suppose further that \( X_1, X_2, \ldots \) is a sequence of points on \( X_0C \) converging to \( C \). Beginning with some \( n \geq N \), one may select pieces \( Y_0Y_i \) corresponding in parameter which are so close to \( X_0X_i \) that for each point \( Y \in Y_0Y_i \) in the plane triangle with the sides of the triangle \( XYC \) the angle at the vertex \( C \) will be less than \( \delta \). For such \( Y \in Y_0Y_i \) the angle between any shortest arc \( YC \) and the direction of \( \mathcal{A} \) at the point \( C \) will be less than \( 3\delta \). Rejecting in advance from the polygons \( A_nC_n \) their ends beyond the points \( Y_i \) for \( n > N \) and requiring that the points \( M_n \) should be chosen on pieces beyond the point \( Y_0 \), we thus satisfy condition 5 of Lemma 15 if \( \delta < \varepsilon/3 \).

The validity of condition 6 of Lemma 15 already follows from conditions 5 and 1.

4. Suppose that \( C \) is not a cusp on \( \mathcal{A} \). Then on the piece \( M_nN_n \) we

![Figure 123](image-url)

Figure 123.
have two simple polygons, before and after the alteration. On Figure 123
the new pieces are depicted by broken curves. In order to compare the
right rotations for the original and altered polygons $L_n, L'_n$, we choose on
them points $M, N$ between which there lies the piece $M_nN_n$ undergoing
alteration.

We may suppose that the points $M, N$ lie in a neighborhood $U$ and that
the alteration of $L_n$ is carried out in a neighborhood of the point $C$ small
in comparison with the distances $CM, CN$.

We join $M$ with $N$ by a new polygon $l_3$ going to the right of $L_n$ and
$L'_n$ (Figure 123). Applying the Gauss-Bonnet theorem to the region bounded
by $l_3 + L_n$ and to the region bounded by $l_3 + L'_n$, we verify that the right
rotations on the closed segment $[M_nN_n]$ for the polygons $L_n$ and $L'_n$ differ
by no more than $\Omega(U) < \Omega(C) + \delta$.

But the right rotation of $L'_n$ at each of the points $M_n, N_n$ and on each
of the open segments $M_nC, CN_n$ is less than $\delta$ in variation. Therefore

$$
s'([M_nN_n]) < |\gamma'_r(C)| + 4\delta < |\gamma'_r([M_nN_n])| + 8\delta
\leq |\gamma_r([M_nN_n])| + \Omega(U) + 8\delta \leq \sigma_r([M_nN_n]) + \Omega(C) + 10\delta,
$$

where the primes denote the rotations on $L'_n$.

Since at the point $C$ the absolute values of the right rotations of $L'_n$
and $\mathcal{S}$ differ, by the construction, by no less than $2\delta$, we have

$$
s'_r(A_nC) + s_r(C)_{\mathcal{S}} + s'_r(CB_n) \leq s_r(L_n) + \Omega(C) + 12\delta,
$$

which for $\delta < \varepsilon/12$ guarantees that inequalities (31) hold.

5. It remains for us to establish inequalities (31) for the more complicated
case when $C$ is a cusp on $\mathcal{S}$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{image}
\caption{Figure 124.}
\end{figure}
In this case the new pieces $MM_nC$ and $CN_nN$ may intersect one another and the right rotation at $C$ for the curve $L'_n$ may be close to $\pi$ or $\theta - \pi$ independently of which of these two numbers is equal to the right rotation of $\mathcal{L}$ at the point $C$. This possibility is illustrated geometrically in Figures 124 a and b.

We choose as earlier points $M, N$ and a polygon $l_b$. Moreover we draw two polygons, $l_1$ from $M$ to $C$ and $l_2$ from $C$ to $N$, going to the left of $L'_n$ and forming with $MM_nC$ and $CN_nN$ at the endpoints $C$ sectors smaller than $3\delta$ and at the endpoints $M$ and $N$ sectors smaller than $\delta$. This construction may be realized, since $CM_n$ and $CN_n$ issue from $C$ in a sector of width $2\delta$ around the common direction of the branches of $\mathcal{L}$, and the sector $2\delta$ may be considered as small in comparison with the complete angle $\theta \equiv 0$ at the point $C$.

Now we will have

$$\sigma_r([M_nN_n]) \geq |\tau_r([M_nN_n])| = |\tau_r(C)_{l_1} + (\tau_r(MN) - \tau_r(l_1 + C + l_2))$$

$$+ (\tau_r(l_1) - \tau_r(MC)) + (\tau_r(l_2) - \tau_r(CN))$$

$$+ \tau_r'(M_n) + \tau_r'(M_nC) + \tau_r'(CN_n) + \tau_r'(N_n)|$$

$$\geq |\tau_r(C)_{l_1}| - \Omega(U) - 5\delta - 5\delta - \delta - \delta - \delta - \delta$$

$$\geq |\tau_r(C)_{l_1}| - \Omega(C) - 15\delta.$$  

We note further that

$$|\tau_r(C)_{l_1}| \geq \min \{\pi - 6\delta; \theta - \pi\},$$

$$\sigma_r(C)_{l_2} = |\tau_r(C)_{l_2}| \leq \max \{\pi; \theta - \pi\}.$$  

Therefore

$$(33) \quad |\tau_r(C)_{l_1}| \geq \sigma_r(C)_{l_2} - \Omega(C) - 6\delta,$$

where $\Omega(C) = |\theta - 2\pi|$. From inequalities (32) and (33) it follows that

$$\sigma_r'(A_nC) + \sigma_r(C)_{l_2} + \sigma_r'(CB_n) \leq \sigma_r(L_n) + 2\Omega(C) + 21\delta.$$

If $21\delta < \varepsilon$ this guarantees that inequalities (31) hold.

Lemma 15 is completely proved.

11. Variation of the rotation of a simple arc and of arcs converging to it from one side.

**Theorem 7.** If the simple arcs $\mathcal{L}$ converge on the right to the simple arc $\mathcal{L} = AB$, there being on $\mathcal{L}$ no singular points at which $\theta = 0$, and the variations of the right rotations $\sigma_r(\mathcal{L}_n)$ are bounded uniformly, then $\mathcal{L}$ is an arc with rotation of bounded variation and

$$(34) \quad \sigma_r(\mathcal{L}) \leq \lim \inf_{n \to \infty} \sigma_r(\mathcal{L}_n).$$
Proof. 1. From Theorem 3, it is sufficient to establish Theorem 7 for the case when all the \( \mathcal{L}_n \) are polygons. By Lemma 14, the arc \( \mathcal{L} \) will have at each point definite directions to the right and left, and therefore also definite rotations on each open arc and at each point.

2. Divide \( \mathcal{L} \) by points \( C_i \) \((i = 0, 1, \ldots, m + 1)\), \( C_0 = A, C_{m+1} = B \) into a finite number of pieces. From Lemma 15, for any \( i \) and beginning with sufficiently large \( n \), one may in arbitrarily small neighborhoods of the points vary the polygons \( \mathcal{L}_n \) so that the new polygons \( \mathcal{L}_n' \) will pass through the points \( A, C_i, \ldots, C_m, B \) and have at these points directions close to \( \lambda \).

On the pieces \( AC_i, C_iC_{i+1}, \ldots, C_mB \) these polygons remain simple, and the sum of the absolute values of their right rotations and of the right rotations of \( \mathcal{L} \) at the points \( C_i \) is bounded by the quantity

\[
\sum_{j=1}^{m} |\tau_j(C_{i}C_{i+1})| + \sum_{i=1}^{m} |\tau_j(C_i)| \leq \sigma_r(\mathcal{L}_n) + \sum_{i=1}^{m} \Omega(C_i) + \varepsilon.
\]

Recalling that on the pieces \( C_iC_{i+1} \) the curves \( \mathcal{L}_n' \) mostly (excluding a small neighborhood of the endpoints) go to the right of \( \mathcal{L} \) and have directions at the ends which are close to \( \mathcal{L} \), we may, by drawing a third curve and making use of the Gauss-Bonnet theorem and of the smallness of the curvature of the region between these curves and the third curve, conclude that for sufficiently large \( n \) we will have the inequality

\[
\sum_{j=1}^{m} |\tau_j(C_{i}C_{i+1})| + \sum_{i=1}^{m} |\tau_j(C_i)| \leq \sigma_r(\mathcal{L}_n) + \sum_{i=1}^{m} \Omega(C_i) + 2\varepsilon.
\]

We note that it was on the transition from (35) to (36) that we made use of the fact that the \( \mathcal{L}_n \) lay to the right of \( \mathcal{L} \).

Passing in (36) on the right to the lower limit with respect to \( n \) and taking account of the arbitrary smallness of \( \varepsilon \), we obtain

\[
\sum_{j=1}^{m} |\tau_j(C_{i}C_{i+1})| + \sum_{i=1}^{m} |\tau_j(C_i)| \leq \mu + \sum_{i=1}^{m} |\tau_j(C_i)| \leq \mu + \Omega(\mathcal{L}),
\]

where for short we have denoted by the letter \( \mu \) the right side of inequality (34). Because of the arbitrary choice of the points \( C_i \) this shows that \( \mathcal{L} \) has rotation of bounded variation.

3. On a curve with rotation of bounded variation the variation is a completely additive function of the arc. Moreover on a simple arc points \( C \) with zero curvature are everywhere dense. Therefore the variation of the rotation \( \sigma_r(\mathcal{L}) \) may be defined as
\[
\sup_{\{C_i\}} \left\{ \sum_{i=0}^{m} |\tau_r(C_i, C_{i+1})| + \sum_{i+1}^{m} |\tau_r(C_i)| \right\},
\]

restricting ourselves to choices of the points \(\{C_i\}\) for which \(\Omega(C_i) = 0\). Therefore it follows from (37) that

\[\sigma_r(\mathcal{Q}) \leq \mu,\]

which proves inequality (34) and Theorem 7 in its entirety.

**Corollary 1.** In Theorem 3 of this chapter it was proved that each simple arc \(\mathcal{Q}\) with rotation of bounded variation may be approximated by polygons \(L_n\) converging to it from the right for which

\[\sigma_r(\mathcal{Q}) \geq \limsup_{n \to \infty} \sigma_r(L_n)\]

It follows from Theorem 7 that for such a sequence

\[\sigma_r(\mathcal{Q}) = \lim_{n \to \infty} \sigma_r(L_n)\]

**Remark.** Under the conditions of Theorem 7 the characteristic \(\tau_r\), and under the conditions of Corollary 1 also \(\sigma^+_r\), \(\tau^-_r\), \(\sigma_r\), as set functions in the domain of variation of the parameter on converging curves, converge weakly to the corresponding functions for the limit curve.

**Corollary 2.** It follows from Theorems 3 and 7 that for an arc with rotation of bounded variation

\[\sigma_r(\mathcal{Q}) = \inf_{\mathcal{Q}_n \to \mathcal{Q}} \liminf_{n \to \infty} \sigma_r(\mathcal{Q}_n),\]

where the greatest lower bound is taken over all possible sequences of simple curves \(\mathcal{Q}_n\) converging to \(\mathcal{Q}\) from the right.

12. Variations of the rotation of a simple arc and of arcs converging to it.

**Theorem 8.** If the simple arcs \(\mathcal{Q}_n\) with right rotations of uniformly bounded variations \(\sigma_r(\mathcal{Q}_n)\) converge to the simple arc \(\mathcal{Q}\), on which there are no points with \(\theta = 0\), then \(\mathcal{Q}\) is an arc with rotation of bounded variation and

\[\sigma_r(\mathcal{Q}) \leq \liminf_{n \to \infty} \sigma_r(\mathcal{Q}_n) + \Omega(\mathcal{Q}).\]

In distinction from Theorem 7, we do not here suppose that \(\mathcal{Q}_n\) converges to \(\mathcal{Q}\) from one side, in fact from the right. As a result (34) is replaced by a quite different inequality (39). The necessity of such a change is easily observed in the simplest examples. It is sufficient, for instance, to
consider on the surface of a cube, as $\mathcal{L}$, two ribs with a common vertex and to approximate $\mathcal{L}$ by polygons $\mathcal{L}_n$ depicted with broken curves on Figure 125.

The proof of Theorem 8 may be carried out same way as in the case of Theorem 7. Only in passing from (35) to (36) we may write in place of (36) just the inequality

$$\sum_{i=0}^{m} \left| \tau_r(C_{i+1}C_i) \right| + \sum_{i=1}^{m} \left| \tau_r(C_i) \right| \leq \sigma_r(\mathcal{L}_n) + \sum \Omega(C_i) + \Omega(\mathcal{L}) + 2\varepsilon,$$

since in the region between the pieces $C_iC_{i+1}$ of the curves $\mathcal{L}, \mathcal{L}'$ and an auxiliary third curve there may fall essential pieces of the curve $\mathcal{L}$, and not just insignificant pieces of small curvature adjacent to $C_i$ and $C_{i+1}$ as was the case in the proof of Theorem 7.

Correspondingly, instead of inequality (34) we obtain inequality (39), which proves Theorem 8.

Remark. An arc $\mathcal{L}$ of bounded variation may be characterized in the following way, not depending on the choice of a concrete side of the curve. The arc $\mathcal{L}$ is decomposed into a finite number of open arcs and points dividing them. For each of these elements one chooses the smaller of the right and left variations. These numbers are added, and one considers the number

$$\sigma(\mathcal{L}) = \inf \sum \min (\sigma_r, \sigma_l),$$

where the greatest lower bound is taken over all possible finite subdivisions.

For the characteristics $\sigma(\mathcal{L})$, apparently, it is always true that under the conditions of Theorem 8 the condition

$$\sigma(\mathcal{L}) \leq \lim \inf \sigma(\mathcal{L}_n)$$

will be satisfied, and also the relation

$$\sigma(\mathcal{L}) = \inf \lim \inf \sigma(\mathcal{L}_n),$$

where the greatest lower bound is taken over all possible sequences of simple arcs $\mathcal{L}_n \to \mathcal{L}$.
5. Possible extensions of the class of curves with rotations of bounded variation.

13. Simple arcs with singular points. According to Definition 1, a curve with rotation of bounded variation can be only a simple arc interior to which and at the endpoints of which there are no singular points where \( \theta = 0 \). We may extend the class of curves in the following way.

**Definition 4.** We include in the class of curves with rotations of bounded variation every simple arc \( \mathcal{L} \) which has definite directions at each of its interior points provided that for any decomposition into a finite number of arcs and open pieces the sum of the absolute values of the right rotations, extended only over those points and pieces whose closures do not contain singular points, remain bounded by some number \( N(\mathcal{L}) \) not depending on the subdivision.

In subsection 6, it was shown by simple examples that for such curves generally speaking there are many results of this chapter which are not valid. In particular, such curves may be nonrectifiable, so that this class decomposes into rectifiable and nonrectifiable curves.

Theorem 6 on pasting apparently remains valid if one pastes regions bounded by *rectifiable* curves satisfying Definition 4. It follows from this assertion, for example, that if from the singular point \( A \) there issue two nonintersecting curves satisfying Definition 4, and one of them is rectifiable close to \( A \), then also the other is rectifiable close to \( A \). (If we cut the space along the first curve and paste into the cut a plane sector with nonzero angle, then \( A \) ceases to be a singular point and the second curve will be rectifiable close to \( A \) as a curve satisfying Definition 1.

For a curve \( \mathcal{L} \) satisfying Definition 4, many of the constructions carried out above for ordinary arcs with rotation of bounded variation may not be realizable. In particular, polygons approximating \( \mathcal{L} \) on one side as constructed in Theorem 3 may not always be made to have common endpoints with \( \mathcal{L} \). However, for these curves there will always exist a one-sided rotation on each open piece, if this is defined in the sense of Lemma 4. There will also exist rotation at separate points, and variation of the rotation.

It is easy to show that if the curves \( \mathcal{L}_n \) of this class converge to a simple arc \( \mathcal{L} \) and one-sided variations of \( \mathcal{L}_n \) are uniformly bounded, then also \( \mathcal{L} \) satisfies the conditions of Definition 4. For the proof it suffices to note that there can only be a finite number of singular points on \( \mathcal{L} \), so that the alteration of the polygons approximating \( \mathcal{L} \) close to these
points leads to a bounded change in the variations of their rotations, and near the other points one may carry out the same constructions as in the proofs of Theorems 6 and 7.

We note once again that points where $\theta = 0$ and the singularities connected with them do not appear on convex surfaces.

14. **Curves made up from simple arcs.** It is natural to extend further the class of curves with rotations of bounded variation in the following way.

**Definition 5 (and 6).** We include in the class of curves with rotation of bounded variation every curve $\mathcal{C}$ which may be decomposed into a finite number of simple pieces each of which is an arc satisfying the conditions of Definition 1 (or 4, respectively).

Depending on whether it is Definition 1 or Definition 4 which we are generalizing, we obtain Definition 5, under which the curves are certainly rectifiable, or Definition 6, under which we admit also certain nonrectifiable curves.

**Singularity connected with cusp points.** For separate simple pieces of a curve satisfying Definition 5 or 6, it makes sense to speak of the one-sided rotation and its variation. The distinction between the right and left sides may be extended in a natural way along the curve from one simple piece to the next. If, moreover, one defines the right and left rotations at the junction points of successive pieces, then the one-sided rotations and their variations extend by additivity to arbitrary pieces along the whole curve. However in the case when the junction point of two simple pieces is a cusp point for the curve in the large, then the two branches issuing from this point may have infinitely many intersections close to the point in question, as in Figure 126c. At such a point $A$ the direction of the branches themselves form two completely defined sector angles $O$ and $\theta(A)$. However in the case of Figure 126c it becomes indefinite which of the two rotations $\pi$, $\pi - \theta$ should be considered the right and which the left rotation of the curve at the point $A$.

In the presence of such singularities of the one-sided rotations and their variations on the curve in the large turn out to not be fully defined numerically. It is natural in this case to speak of the upper and lower
rotations $\tau_r$, $\tau_{(-r)}$, and of the upper and lower variations $\sigma_r$, $\sigma_{(-r)}$, choosing at each such point correspondingly the largest or smallest of the two possible values of the rotation or variation.

This kind of singularity may also arise on a convex surface.\footnote{This was not mentioned in the book [42].}

Consideration of Definitions 4, 5 and 6 is interesting because the limiting passage from curves satisfying Definition 1 may be carried to more complicated curves.

The following assertions are apparently true.

1. If the curves $\mathcal{L}_n$ converge to the curve $\mathcal{L}$, with the $\mathcal{L}_n$ satisfying Definition 6, and if $\sigma_{(-r)}(\mathcal{L}_n) \leq C$, then also $\mathcal{L}$ belongs to this same class of curves and

$$\sigma_r(\mathcal{L}) \leq \liminf_{n \to \infty} \sigma_{(-r)}(\mathcal{L}_n) + \Omega(\mathcal{L}),$$

where $\Omega(L)$ is defined taking account of the multiplicity of points of $\mathcal{L}$.

2. If moreover, there are no points on $\mathcal{L}$ where $\theta = 0$, then, beginning with some $n$, all the $\mathcal{L}_n$ are rectifiable and their lengths are uniformly bounded.

Finally we note that we may also speak of the right and left rotations not of arcs but of closed curves under the condition that along the curve the distinction between the right and left rotations is preserved. This last does not hold for example for the closed center line of the Möbius strip.
Complements to Chapter II

On Various Definitions of the Angle

We consider a series of possible definitions of angle. All of them characterize the rapidity of departure from one another of curves issuing from a common point, and in the case of regular curves in Euclidean (or Riemannian) space lead, as a rule, to the usual values of the angle. The situation is different in more complicated spaces. In Chapter II we needed only two concepts, the upper angle and the lower strong angle between shortest arcs. The material presented below will show why preference was given to these two definitions. Further, these materials may be used for other kinds of generalized expositions of the theory.

1. Triangle on a $K$-plane. Suppose in an arbitrary metric space that there issue from the point $O$ two curves $L = X(t)$ and $M = Y(s)$. We select points $X$ and $Y$ on them distinct from $O$. Suppose further that

\[ \rho(O, X) = x, \quad \rho(O, Y) = y, \quad \rho(X, Y) = z. \]

In subsection 4 of Chapter I, in introducing the concept of the angle between $L$ and $M$, we made use of the auxiliary angle $\gamma(X, Y)$, constructing on the plane a triangle $T_0$ with sides $x, y$ and $z$ and considering in it the angle $\gamma$ opposite the side $z$. But we could have constructed instead of $T_0$ a triangle $T_K$ with the sides $x, y, z$ on a surface with an arbitrarily fixed constant curvature $K$. We shall call such a surface a $K$-plane.\(^1\) Here the angle $\gamma_K(X, Y)$ opposite the side $z$ will be quite different from $\gamma(X, Y) = \gamma_0(X, Y)$.

As is known from differential geometry, for the angles $\alpha_K, \beta_K, \gamma_K$ and $\alpha_0, \beta_0, \gamma_0$ of the triangles $T_K$ and $T_0$ there is the equation

\[ (\alpha_K - \alpha_0) + (\beta_K - \beta_0) + (\gamma_K - \gamma_0) = \sigma K, \]

where $\sigma$ is the area of the triangle $T_K$. We need to add to this that all three differences in parentheses on the left side of (2) are either simultaneously equal to zero (for $K = 0$ or $\sigma = 0$), or have the same sign as the quantity $K$. This last elementary assertion may be verified on the example of the angles $\gamma_K$ and $\gamma_0$, starting from the explicit expressions for the

\[^1\text{If } K \leq 0 \text{ the triangle } T_K \text{ exists, since } x, y, \text{ and } z \text{ satisfy the triangle inequality. If } K = k^2 > 0 \text{ it may be constructed under the conditions } kx, ky, kz < \pi, kx + ky + kz \leq 2\pi. \text{ These conditions will be supposed satisfied when we speak of constructing a triangle on the corresponding } K\text{-plane.}\]
cosines of these angles. If $K = -k^2 < 0$ and $x$ and $y$ are fixed and arbitrary, then for all $z$ in the interval $|x - y| \leq z \leq x + y$ we have

$$\frac{\cosh kx \cosh ky - \cosh kz}{\sinh kx \sinh ky} = \frac{x^2 + y^2 - z^2}{2xy} \geq 0.$$  

For $K = k^2 > 0$ and arbitrary fixed $0 < kx, ky < \pi$, for all $z$ in the interval $|x - y| \leq z \leq x + y$ we have

$$\frac{\cos kz - \cos kx \cos ky}{\sin kx \sin ky} - \frac{x^2 + y^2 - z^2}{2xy} \leq 0.$$  

Equality here is attained only at the endpoints of the interval of variation of $z$. Therefore

$$|\gamma_K - \gamma_0| \leq \sigma |K|.$$  

Thus it follows that if we are interested in the limiting values of the angles $\gamma_K(X, Y)$ for sequences of points $X, Y$ for which the area $\sigma(T_K) \to 0$, then it makes no difference whether we consider the angles $\gamma_K(X, Y)$ or $\gamma_0(X, Y)$.

**Remark.** Lemma 1 of Chapter I remains valid for the angles $\gamma_K$:

$$\cos \gamma_K = \frac{y - z}{x} + \varepsilon,$$

where $\varepsilon \to 0$ as $x/y \to 0$, with the additional requirement that $\sigma(T_K) \to 0$. For $K > 0$ the condition $\sigma \to 0$ follows from $x/y \to 0$, since in this case $y$ is supposed bounded ($y \sqrt{K} < \pi$). For $K < 0$, for $\sigma \to 0$ it suffices that not only $x/y$ but also $x \to 0$.

2. **Upper and lower angles.** The lower, upper and ordinary angles $\alpha_-, \bar{\alpha}$ and $\alpha$ between $L$ and $M$ were defined in Chapter II respectively as the lower, upper and ordinary limits of the angles $\gamma(X, Y)$ as $X, Y \to O$, $X \in L$, $Y \in M$, $X \bot O$, $Y \bot O$. Evidently $0 \leq \alpha_- \leq \bar{\alpha} \leq \pi$. The angle $\alpha$ exists when $\alpha_- = \bar{\alpha}$.

The properties of the angle $\bar{\alpha}$ were considered in § 1, Chapter I.

The essential difference between the lower angle $\alpha_-$ and the upper angle $\bar{\alpha}$ is connected with the asymmetry of the basic triangle inequality. For lower angles assertions of the type of the theorems of Chapter II do not hold. In connection with this, in Chapter II the more complicated concept $\alpha_{(-\varepsilon)}$ was investigated. We restrict ourselves to an example connected with Theorems 5 and 6.

**Example.** Suppose that in the plane curvilinear triangle depicted in Figure 127 the lengths of the convex curves $AB$, $AC$ and of the straight
side $BC$ are equal to $l$, and angle $\phi < \pi/3$. We construct an abstract space, consisting of the three threads $AB = BC = CA = l$ and an infinite number of other threads joining pairwise the points $X \in AB$ and $Y \in AC$ and having the same lengths as the corresponding shortest arc $XY$ in Figure 127. Along the lengths of the curves in this abstract space we introduce an intrinsic metric. We may verify that in the resulting space $ABC$ is a triangle. In it at the vertex $A$

$$\alpha_0 = \phi, \quad \alpha_0 = \frac{\pi}{3}, \quad \alpha_0 - \alpha_0 < 0.$$ 

At the same time for any triangle $AXY$

$$\delta_-(AXY) = (\phi + \pi + \pi) - \pi = \pi + \phi.$$ 

Thus for the quantity

$$\nu_\Delta = \inf_{X \in AB} \inf_{Y \in AC} \delta_-(AXY)$$

we do not have a relation of the type of Theorem 6.²

$$\alpha_0 - \alpha_0 \neq \nu_\Delta.$$ 

3. **Angle in the weak sense.** The limit of the angles $\gamma(X,Y)$ may be considered under additional restrictions of the possible situations of the points $X, Y$. Sometimes it is comparatively easy to follows the value of $\gamma(X,Y)$ under the condition that the ratio of the distances $x$ and $y$ of the points $X$ and $Y$ from $O$ remains within limits:

$$0 < a \leq \frac{x}{y} \leq b < \infty.$$ 

We shall call the upper weak angle and the lower weak angle the following limits, which always exist and do not depend on $a$ and $b$:

$$\bar{\alpha}_w = \lim_{a \to 0} \lim_{b \to \infty} \sup_{X,Y \to 0} \gamma(X,Y),$$

$$\bar{\alpha}_{(\bar{w})} = \lim_{a \to 0} \lim_{b \to \infty} \inf_{X,Y \to 0} \gamma(X,Y).$$

As before we consider only points $X \in L, \ Y \in M, \ X \neq O, \ Y \neq O$. 

² This example answers the question set in [13], the footnote on page 8. However it is not clear whether one can find an analogous example in a space which is a two-dimensional manifold.
Suppose that \( \alpha_{(-)W} = \overline{\alpha}_w \), i.e. for any \( 0 < a \leq b < \infty \) there exists the limit

\[
\alpha_w = \lim_{0 < a \leq x/y \leq b < \infty} \gamma(X, Y),
\]

which thus will not depend on the choice of \( a \) and \( b \). Then its value \( \alpha_w = \alpha_{(-)W} = \overline{\alpha}_w \) is called the weak angle, or the angle in the weak sense.

We may further consider, so to speak, the "weakest" upper, lower, and simple angles \( \overline{\alpha}_{WW}, \alpha_{(-)WW}, \alpha_{WW} \), imposing the more rigid condition \( x = y \). Evidently

\[
0 \leq \alpha_\leq \alpha_{(-)W} \leq \alpha_{(-)WW} \leq \overline{\alpha}_{WW} \leq \overline{\alpha}_w \leq \overline{\alpha} \leq \pi.
\]

**Theorem 1.** If each of two curves has a definite direction, then the weak upper angle between them is equal to the upper angle:

\[
\overline{\alpha}_w = \overline{\alpha}.
\]

**Proof.** Since always \( \overline{\alpha}_w \leq \overline{\alpha} \), it suffices to show that under the conditions of the theorem \( \overline{\alpha}_w \leq \overline{\alpha}_w \).

We choose on the curves \( L \) and \( M \) in question points \( X_n, Y_n \) with \( \gamma(X_n, Y_n) \rightarrow \overline{\alpha} \) and converging to \( O \). If in addition \( 0 < a \leq x_n/y_n \leq b < \infty \), then \( \overline{\alpha}_w \leq \overline{\alpha}_w \). Suppose that \( X_n/Y_n \rightarrow O \) (if \( X_n/Y_n \rightarrow \infty \), we change the names of \( x \) and \( y \)). On \( M \) we mark points \( Y'_n \) for which \( x_n/y_n = a \), where \( \overline{\alpha}_w > 0 \) is an arbitrarily small number. We construct on the plane the corresponding triangle \( OXY' \) with sides \( x_n, y'_n, z'_n \). To its side \( OY' \) we adjoin another triangle \( OY'Y \) with the sides \( y'_n, y_n, Y'_nY_n \), as in Figure 128.

Now it is easy to explain why \( \overline{\alpha}_w \leq \overline{\alpha}_w \). The angle at the vertex \( O \) in the plane triangle \( OXY' \) cannot for large \( n \) essentially exceed \( \overline{\alpha}_w \). The angle \( Y'OY \) is small, since the curve \( M \) has a direction. Finally, \( z_n = X_nY_n \leq X_nY'_n + Y'_nY_n \). But because of the smallness of \( a = x_n/y_n \) even on rectification of the side \( YY'X \) in the plane quadrilateral \( OYY'X \) the angle at the vertex \( O \) cannot essentially increase, i.e., \( \gamma(X_n, Y_n) \), and therefore also \( \overline{\alpha}_w \) cannot essentially exceed \( \overline{\alpha}_w \).

Let us make this more precise. By the choice of \( X_n, Y_n \) and Lemma 1 of Chapter II,

\[
\cos \overline{\alpha}_w = \lim_{n \to \infty} \frac{y_n - z_n}{x_n}.
\]
Since $M$ has a direction, the angle $\gamma(Y_n, Y'_n) \to 0$, i.e., $(y_n - Y'_n)/y'_n \to 1$ or $Y'_n Y_n = y_n - y'_n + \varepsilon_n y_n$, where $\varepsilon_n \to 0$. Finally,

$$z_n \leq X_n Y'_n + Y'_n Y_n = z'_n + y_n - y'_n + \varepsilon_n (x_n/\alpha),$$

so that

$$\frac{y_n - z_n}{x_n} \leq \frac{y'_n - z'_n}{x_n} - \frac{\varepsilon_n}{\alpha}.$$ 

Thus again using Lemma 1 of Chapter II, this time with the sharpening (6) of subsection 3 of Chapter II, we have:

$$\cos \bar{\alpha} \geq \limsup \frac{y'_n - z'_n}{x_n} \geq \limsup_{n \to \infty} \left[ \cos \gamma(X_n, Y'_n) - \frac{1}{2} \frac{x_n}{y'_n} \right]$$

$$\geq \liminf_{x \to 0} \cos \gamma(X, Y) - \frac{a}{2} = \cos \left[ \limsup_{x \to 0} \gamma(X, Y) \right] - \frac{a}{2}.$$ 

But $a > 0$ may be taken arbitrarily small and $b$ arbitrarily large. Therefore $\cos \bar{\alpha} \geq \cos \bar{\alpha}_w$ and $\bar{\alpha} \leq \bar{\alpha}_w$. The theorem is proved.

**Remarks.** 1) Theorem 1 essentially complements Theorem 4 of Chapter II, establishing the still greater stability of the upper angle.

2) For the lower angle an assertion analogous to the theorem just proved does not hold. *Example*. Compare the plane sector bounded by the arcs $L$ and $M$ with the acute angle $\phi$ in a conical trough (Fig. 129). We join individual points of its boundary in space by segments $A_iB_i$, $A_2B_2$, ..., and suppose that the angle of inclination of these segments to the straight line $L$ tends to zero as they approach the vertex $O$. In the metric space which the cone along with the adjoined threads $A_iB_i$ represents, the arcs $L$ and $M$ are shortest arcs. The weak angle between them exists and is equal to the complete angle $\phi$ of the sector, and the lower angle may be obtained starting from the sequence of points $A_iB_i$. It is equal to the space angle between $L$ and $M$, which is less than $\phi$. In this example $\alpha_- < \alpha_{-w} = \bar{\alpha}_w = \bar{\alpha} = \phi$.

3) If the ordinary angle exists, then the weak angle exists and coincides with it. But curves may form a weak angle but not an ordinary one. This is shown by the last example. Here are other examples. Suppose that the plane spiral $L$ (Figure 130) forms infinitely many loops as it
ON VARIOUS DEFINITIONS OF THE ANGLE

approaches the center $O$. Each $i$th loop is a piece of a logarithmic spiral on which the segment $OX$ forms with $L$ the angle $\alpha_i$. If $\alpha_i \to 0$ as $i \to \infty$, then as is easily verified, the spiral $L$ forms at $O$ with itself the weak angle $\alpha_w = 0$. But in the ordinary sense $L$ does not have a direction at $O$. The curve given on the plane by the equation $y = x \sin \ln |\ln x|$ has the same property at the point $(0,0)$.

4) The concept of weak angle is used for example in the paper [31].

5) Theorem 1 generally speaking ceases to be true if one of the curves does not have a definite direction. **Example.** We erect perpendiculars at the points of the plane spiral $L$ of Figure 130. In the metric space consisting of the plane in which the spiral $L$ lies and the resulting cylindrical surface, we consider the angle at the point $O$ between $L$ and the perpendicular $M$. In this example, as is easily verified,

$$\pi/2 = \alpha_- = \alpha_{(-)_E} = \alpha_w < \bar{\alpha} = \pi.$$

4. **Extended angle.** One may consider the limit of the angles $\gamma(X, Y)$ under widened possibilities for the positions of $X$ and $Y$. Suppose that $X$ and $Y$ do not necessarily lie on the curves $L$ and $M$, but rather that as they approach zero the distances from $X$ and $Y$ to $L$ and $M$ respectively go down faster than the distance from $O$:

$$\frac{\rho(X, L)}{\rho(X, O)} \to 0, \quad \frac{\rho(Y, M)}{\rho(Y, O)} \to 0.$$

The upper and lower limits of the angles $\gamma(X, Y)$ for all possible such sequences $X, Y \to O$ will be called the upper and lower extended angles $\alpha_E, \alpha_{(-)_E}$. If $\alpha_{(-)_E} = \alpha_E$, their common value is called the extended angle between $L$ and $M$. Obviously, it is always true that

$$0 \leq \alpha_{(-)_E} \leq \alpha_- \leq \bar{\alpha} \leq \alpha_E \leq \pi.$$

**Remark.** We give examples when $\alpha_{(-)_E} < \alpha_-$ or $\bar{\alpha} < \alpha_E$.

1) Suppose that $L$ and $M$ are rays issuing from $O$ on the plane, forming
an acute angle \( \phi \), and \( N \) is a curve lying in the same plane tangent to \( M \) from within at the point \( O \). From the point \( Y_1 \) on the curve \( N \) we drop a perpendicular \( Y_1A_1 \) onto \( M \). We choose \( X_1 \in L \) so that \( X_1Y_1 + Y_1A_1 = X_1O + OA_1 \). Then we choose the point \( Y_2 \in N \) very much closer to \( O \) than \( X_1 \) and \( Y_1 \). We repeat this construction as in Figure 131. In the intrinsic geometry of the figure consisting only of the threads \( L, M, N, A_1Y_1, A_2Y_2, \ldots \) we will have \( \alpha_{-\infty} \leq \phi < \pi - \alpha_- \) for the angle between \( L \) and \( M \).

2) Consider the plane sector \( LOM \) with acute angle \( \phi \) and a convex arc \( N \) lying in the same plane and tangent to \( M \) from outside at the point \( O \), as in Figure 132. From the point \( Y_1 \in N \) we drop a perpendicular \( Y_1A_1 \) onto \( M \). On \( L \) we choose a point \( X_1 \) so close to \( O \) that \( Y_1A_1 + A_1X_1 > Y_1O + OX_1 \). This is possible since by the convexity of the arc \( Y_1O < Y_1A_1 + A_1O \). Then we choose a point \( Y_2 \in N \) very much closer to \( O \), and repeat the construction, and so forth. In the intrinsic geometry of the figure made up of the plane sector \( LOM \) and the threads \( N, A_1Y_1, A_2Y_2, \ldots \) for the angle between \( L \) and \( M \) we will have

\[
\bar{\alpha} = \phi < \pi = \bar{\alpha}_B.
\]

5. Nonlocal characteristics of the angle between shortest arcs. In the definition of the angle in the strong sense (§ 2 of Chapter II), we considered, for the shortest arcs \( L = OX_o, M = OY_o \) the lower and upper limits \( \alpha_{-\infty}, \bar{\alpha}_s \) of the angles \( \gamma(X, Y) \), taken for all possible sequences of points \( X_n, Y_n \) for which

- \( X_n \in L, Y_n \in M \), \( X_n \nless O, Y_n \nless O \), \( X_n \rightarrow O \) or \( Y_n \rightarrow O \);

- if \( X_n \rightarrow O \) there exist shortest arcs \( X_nY_n \) converging to a piece of the shortest arc \( M \), and if \( Y_n \rightarrow O \) to a piece of \( L \) (we suppose that at least one such sequence \( X_n, Y_n \) exists).

The upper weak angle coincides with \( \bar{\alpha} \), and the lower \( \alpha_{-\infty} \) is a nonlocal characteristic of the angle.
This last insufficiency is also suffered by the always existing quantity \( \alpha_\infty \), defined as the lower limit of the angles \( \gamma(X, Y) \), taken over all possible sequences \( X_n, Y_n \) satisfying condition a) without condition b).

To a certain extent the abovementioned insufficiency is excluded if one turns from \( \alpha_{(-\infty)} \) to the characteristic \( \alpha_{(-\infty)} \), defined as the greatest lower bound of the values of \( \alpha_{(-\infty)} \) for all possible pairs of shortest arcs, which on arbitrarily small initial segments coincide with \( L \) and \( M \). This characteristic was used in [5].

Analogously one may define \( \alpha_\infty \) as the lower bound of \( \alpha_\infty \) for various extensions of the initial segments of \( L \) and \( M \).

The definitions of the quantities \( \tilde{\alpha}, \bar{\alpha}, \tilde{\alpha} \) are obtained by a replacement of the lower limit by the upper limit and of the greatest lower bound by the least upper bound. But by Theorem 3 of Chapter II all these coincide with \( \bar{\alpha} \).

**Remarks.** 1) For the angle \( \alpha_\infty \) an assertion holds which is similar to Lemma 3 of Chapter II. In fact,

\[
(9) \quad \left( \frac{\partial z}{\partial x} \right)_L \leq \cos \alpha_\infty.
\]

2) Assertions of the type of Lemmas 5, 6 and 7 of Chapter II and Theorem 6 of Chapter II are also valid. But this time

\[
(10) \quad \alpha_\infty - \alpha_0 \geq \nu(\alpha_\infty A),
\]

where

\[
(11) \quad \nu(\alpha_\infty A) = \inf_{X \in AB} \sup_{Y \in AC} \left[ \frac{\partial z}{\partial x} (AXY) \right].
\]

3) All these assertions are proved analogously, and even somewhat more simply than the corresponding theorems of Chapter II. However, the angle \( \alpha_\infty \) cannot replace the angle \( \alpha_{(-\infty)} \) in the construction of the theory of two-dimensional manifolds of bounded curvature. In these spaces there exists an angle in the strong sense between shortest arcs, but the characteristic \( \alpha_\infty \) may fail to coincide with this angle.

4) We give a simple example when \( \alpha_\infty < \alpha_{(-\infty)} \). Consider on the sphere two shortest arcs \( L \) and \( M \) issuing from the point \( O \). Suppose that they form at \( O \) an acute angle \( \phi \), with the opposite ends of the shortest arcs coinciding and lying at a point diametrically opposite to \( O \). For such shortest arcs \( \alpha_\infty = 0 < \phi = \alpha_{(-\infty)} \).

If in this last example we somewhat shorten \( L \) and \( M \), we will have an example in which \( \alpha_\infty = 0 < \phi = \alpha_\infty \).
5) In the case of the so-called manifolds of nonpositive curvature, in which all the excess $\delta \leq 0$ or of "negative curvature not greater than $K$" (see [13], § 4) for shortest arcs always $\alpha_\ominus = \bar{\alpha}$, so that there exists an angle in this more extended sense.

6. Relations of the various definitions of angle.

**Lemma.** For the shortest arcs $L$ and $M$ in a locally compact space with intrinsic metric, always $\alpha_{(\ominus)E} \leq \alpha_{(\ominus)S}$.

**Proof.** Consider a sequence $X_n, Y_n$ for which $\gamma(X_n, Y_n) \rightarrow \alpha_{(\ominus)S}, \rho(O, X_n) = x_n \rightarrow 0, X_n Y_n \rightarrow \overline{OY} \subset M$. If moreover $\rho(O, Y_n) = y_n \rightarrow 0$ then evidently $\alpha_{(\ominus)E} \leq \alpha_{(\ominus)S}$. Suppose that $y_n \geq a > 0$. Then on the shortest arcs $X_n Y_n$ one may select respectively points $Y'_{n'}$ such that $\rho(Y_{n'}, M)$ and $x_n$ decrease faster than $y_{n'}' = \rho(O, Y_{n'})$. From the triangle $O Y_n Y'_{n'}$ we have

$$y_{n'}' + (z_n - z_{n'}') \leq y_n,$$

where $z_n = \rho(X_n, Y_n)$, $z_{n'}' = \rho(X_n, Y'_{n'})$. Therefore

$$\cos \alpha_{(\ominus)S} = \lim \frac{y_n - z_n}{x_n} \leq \limsup \frac{y_{n'}' - z_{n'}'}{x_n} \leq \cos \alpha_{(\ominus)E}.$$

This proves the lemma in question. Analogously one may verify that $\alpha_{(\ominus)E} \leq \alpha_{(\ominus)S}$. Directly from the definitions and also from Theorem 3 of Chapter II and Theorem 1 of this supplement and the last lemma it results that the following theorem is valid.

**Theorem 2.** In a locally compact space with an intrinsic metric, for the various characteristics of the angle between two shortest arcs the following relations are valid:

$$0 \leq \alpha_\ominus \leq \alpha_\ominus \leq \alpha_{(\ominus)S} \leq \alpha_{(\ominus)W} \leq \alpha_{(\ominus)WW} \leq \bar{\alpha}_{WW} \leq \bar{\alpha}_W = \bar{\alpha}_{E} \leq \pi.$$

For each sign "$\leq$" in the chain of relations (12) one may present an example in which the strict inequality is realized. In subsection 3 the example for $\alpha_\ominus < \alpha_{(\ominus)W}$ was given, in subsection 4 examples for $\alpha_{(\ominus)E} < \alpha_\ominus$, $\bar{\alpha} < \bar{\alpha}_E$, and in subsection 5 $\alpha_\ominus < \alpha_{(\ominus)S}$, $\alpha_\ominus < \alpha_\ominus$. We shall give further an example in which $\alpha_{(\ominus)S} < \alpha_\ominus$.

In the plane sector $LOM$ with acute angle $\phi$ we mark points $A_i \rightarrow O$ which approach the side $L$ faster than $O$, as in Figure 133. Along cuts along the segments $YA_i$ we paste high, twice-covered partitions. On the resulting surface the angle at the point $O$ between $L$ and $M$ will satisfy
\[ \phi = \alpha_{\gamma \beta} < \alpha_\gamma - \pi. \]

The construction of the missing examples is left to the reader.

7. *Comparison with a triangle on a K-plane.* The excess of a triangle may be measured by the difference of the sum of its angles (in one or another of the definitions) from the sum of the angles of the triangle with sides of the same length on a K-plane. For such "relative" excesses \( \delta_k(T), \delta_{\gamma < \beta k}(T), \delta_{\gamma > \beta k}(T) \) it is possible to define the corresponding quantities \( \nu \) analogously to the definitions (20), (27) of Chapter II or (12) of this supplement. We shall denote them by the supplementary index \( K \).

For angles \( \gamma_k \) on a K-plane Lemmas 4, 5 and 6 of Chapter II are valid with the following alterations.

1. In Lemma 4 formula (12) is replaced by the following:

\[ \frac{\Delta z}{\Delta x} = \cos \xi_k + \frac{x}{\lambda} \frac{\Delta \gamma_k}{\Delta x} \sin \xi_k + \varepsilon, \]

where

\[ \lambda = \begin{cases} 
\frac{kx}{\sin kx} & \text{if } K = k^2 > 0, \\
1 & \text{if } K = 0, \\
\frac{kx}{\sinh kx} & \text{if } K = -k^2 < 0.
\end{cases} \]

For the proof it suffices only in the infinitesimal discussion of Chapter II to have in mind a construction on a K-plane and to replace formula (13) of Chapter II by the law of sines on a K-plane, i.e., one of the three expressions

\[ kl = \sin kx \sin \Delta \gamma_k, \quad l = x \sin \Delta \gamma_k, \quad kl = \sinh kx \sin \Delta \gamma_k. \]

We note that it follows from (14) that for any \( x \), if \( K < 0 \) and all \( 0 < x < \pi/k \) the quantity \( \lambda \) has a positive minimum for \( K = -k^2 < 0 \).

2. In Lemma 5 inequalities (14) and (15) are replaced by

\[ \left( \frac{\partial \gamma_k}{\partial x} \right)_{ll} \geq \frac{\cos \xi - \cos \xi_k}{\sin \xi_k} \cdot \frac{\lambda}{x}, \]

*Figure 133.*
COMPLEMENTS TO CHAPTER II

\[
\frac{\partial \gamma_K}{\partial x} \leq \frac{\cos \xi - \cos \xi_K}{\sin \xi} \cdot \frac{\lambda}{x},
\]

where \( \lambda \) is determined by (14).

3. In Lemma 6 of Chapter II the angles \( \gamma \) and \( \xi_0 \) are replaced by \( \gamma_K \) and \( \xi_K \). Moreover, the constant \( M \) depends this time not only on \( \varepsilon \) but also on the minimal value of \( \lambda \), which in its turn depends on \( K \) and the upper estimate of the diameter of the triangle.

4. Theorems 4 and 5 of Chapter II take this time the following form:

\[
\bar{\alpha} - \alpha_K \leq \bar{\nu}_{KA},
\]
\[
\alpha_\Kbar - \alpha_K \leq \nu_{\Kbar KA},
\]
\[
\alpha_{\Kd} - \alpha_K \geq \nu_{\Kd SKA}.
\]

The proofs remain as before.
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